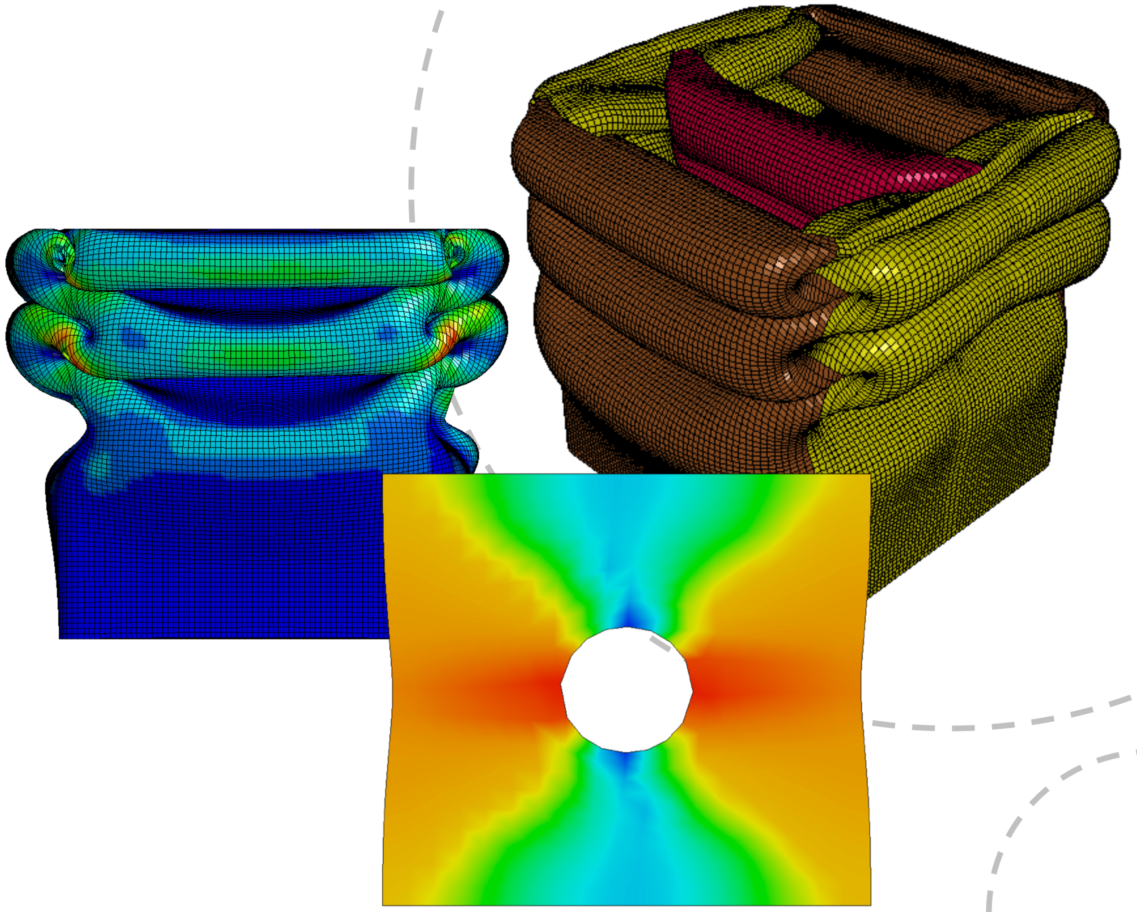


Plasticity theory

KT8306

D. Morin

Learning goals and topics

The course aims at giving the participants a comprehensive introduction to formulation and implementation of constitutive models for elastic-plastic materials.

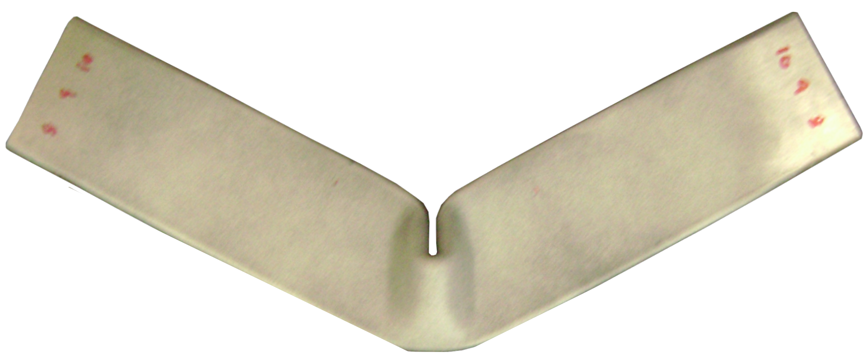
Main topics:

- Plasticity
- Visco-plasticity
- Stress update algorithms
- Damage mechanics
- Hypo/Hyper-elastic-plastic plasticity

Plasticity theory

Introduction

Plasticity describes the deformation of a material experiencing irreversible changes of shape in response to applied forces.

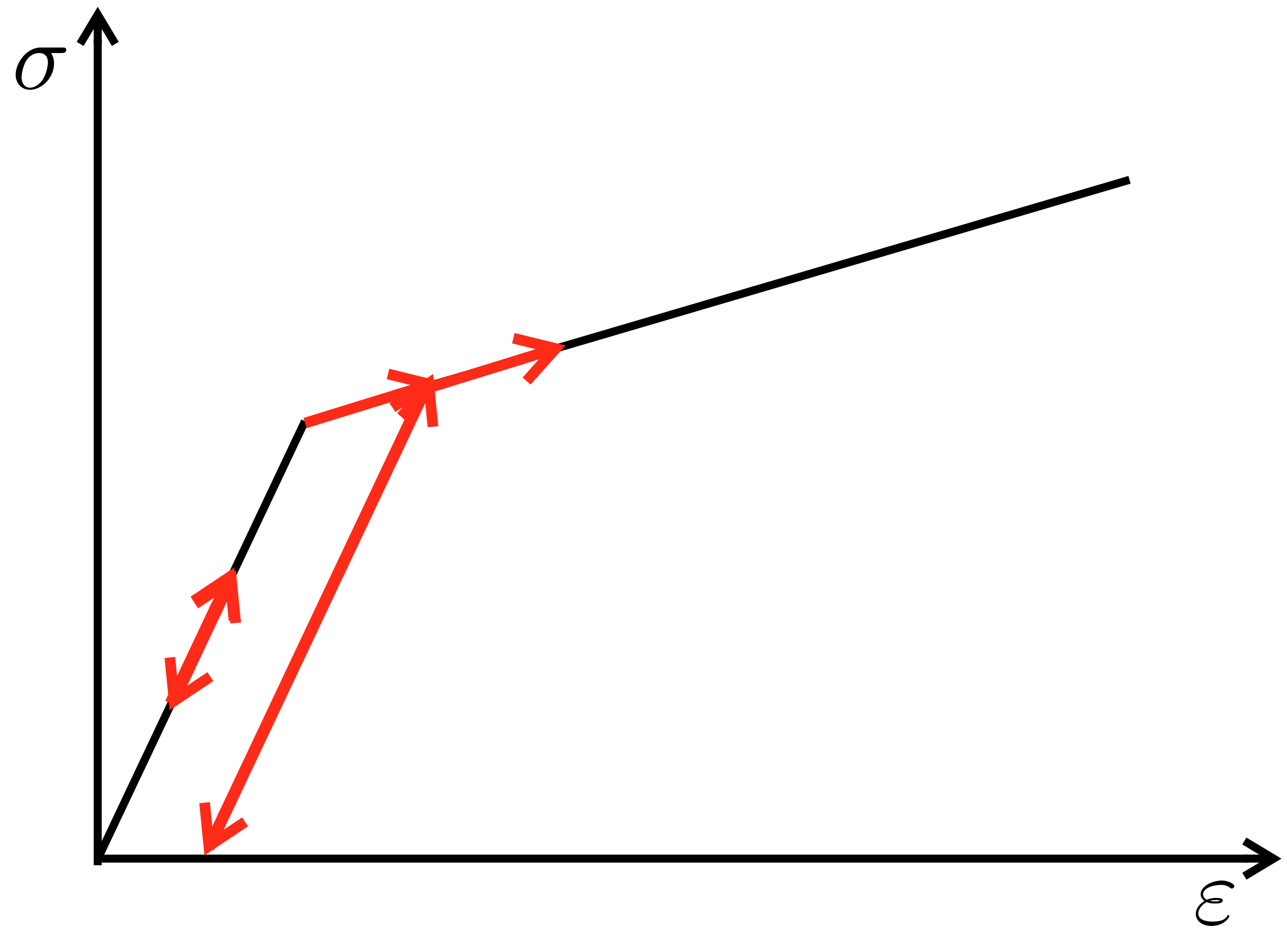


Introduction

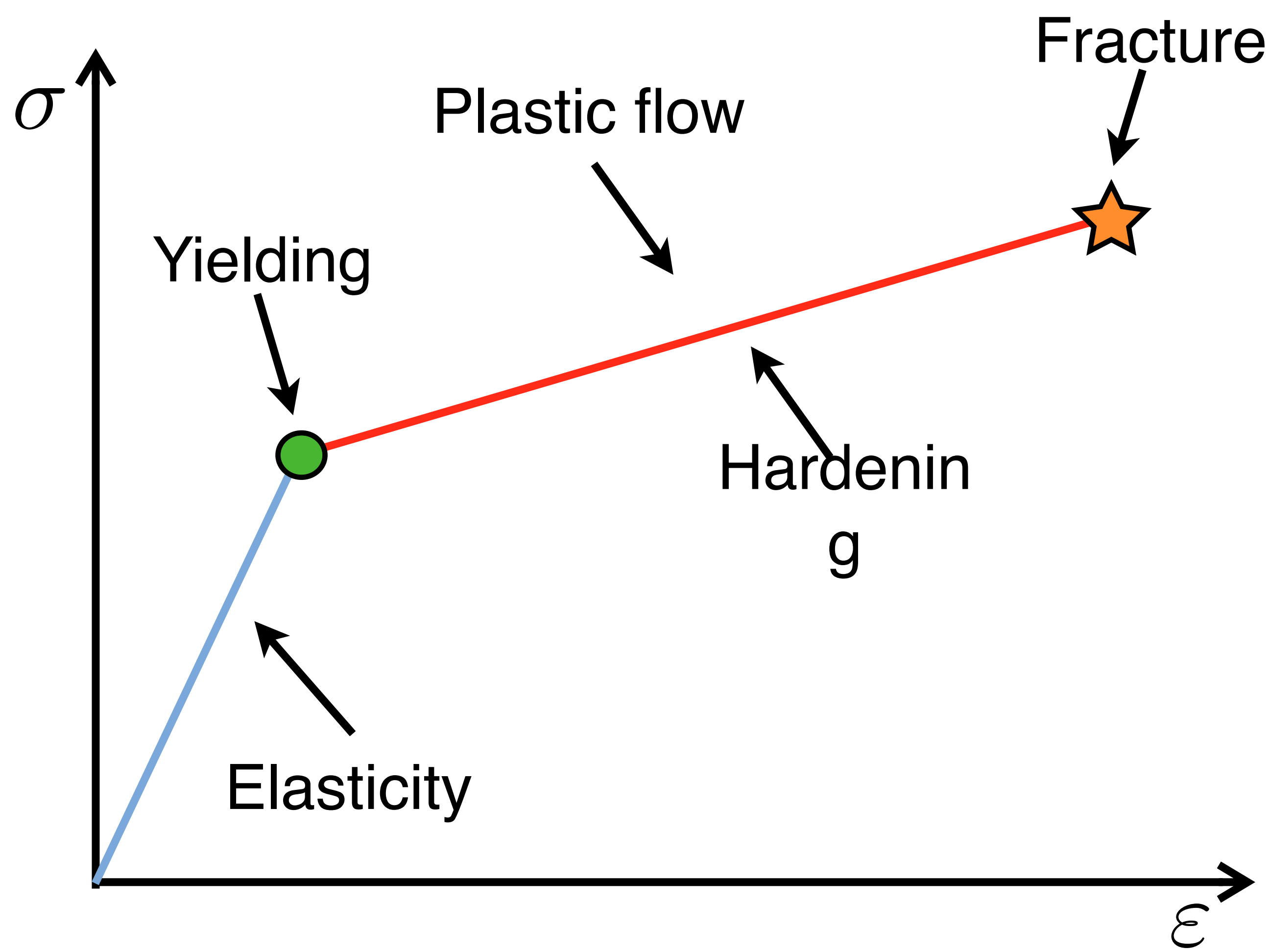
Plasticity describes the deformation of a material experiencing irreversible changes of shape in response to applied forces.

- Irreversible deformation
- Path dependence
- Energy dissipation
- Rate independence

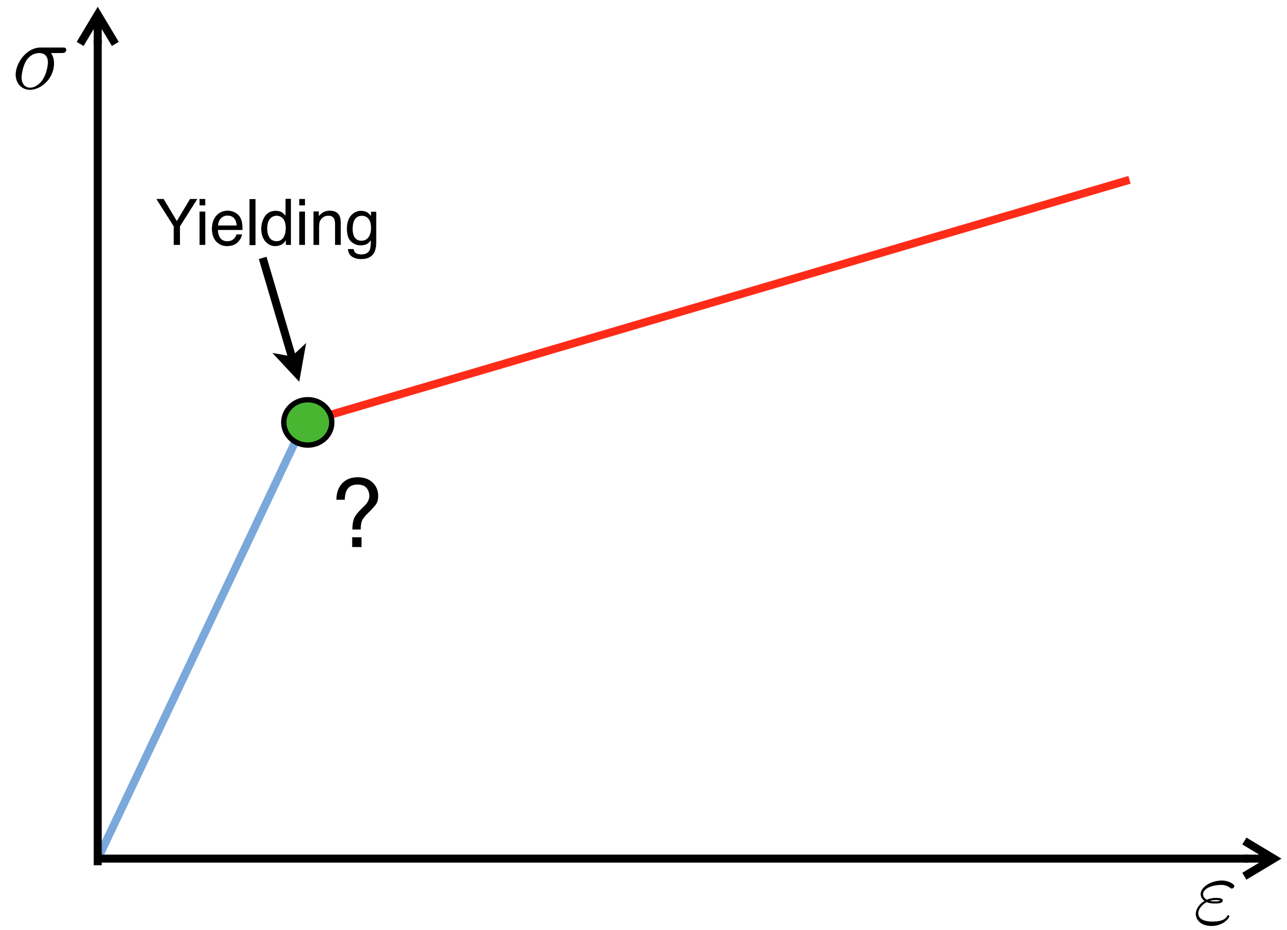
Introduction



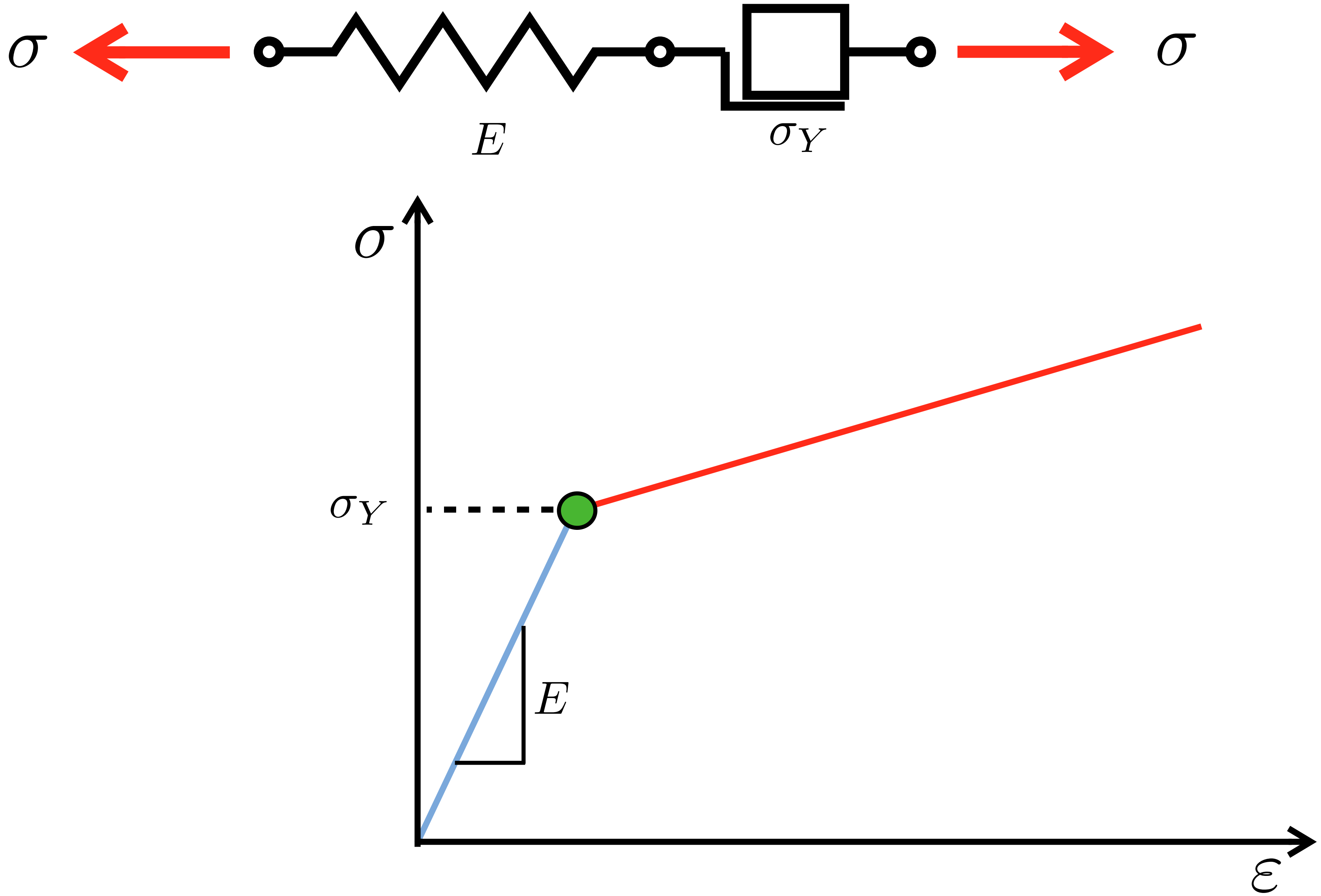
Introduction



Yield criterion

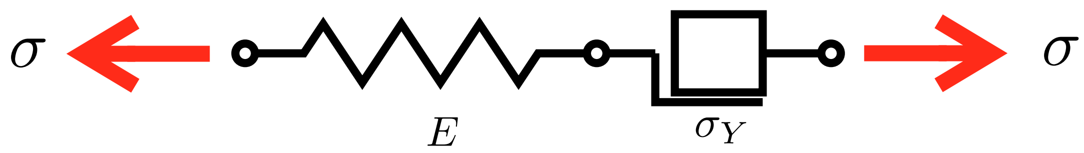


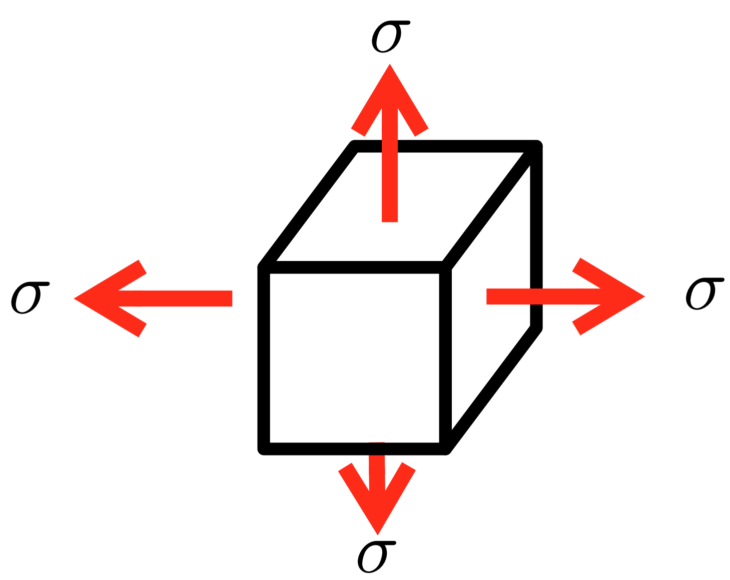
Yield criterion



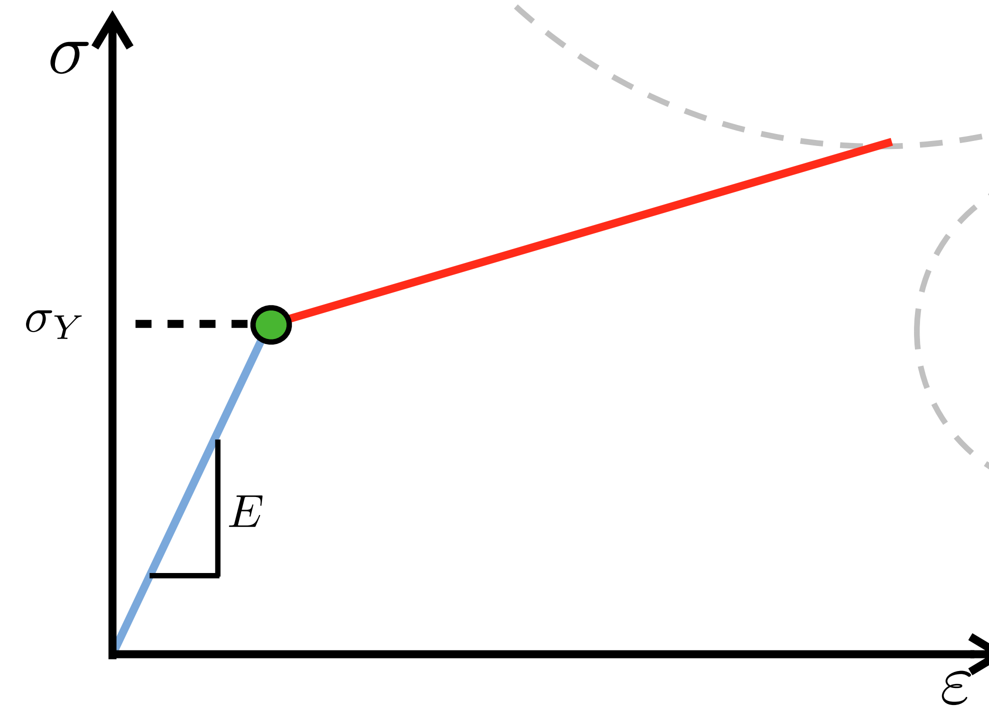
$$f = |\sigma| - \sigma_Y$$

Yield criterion





$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$



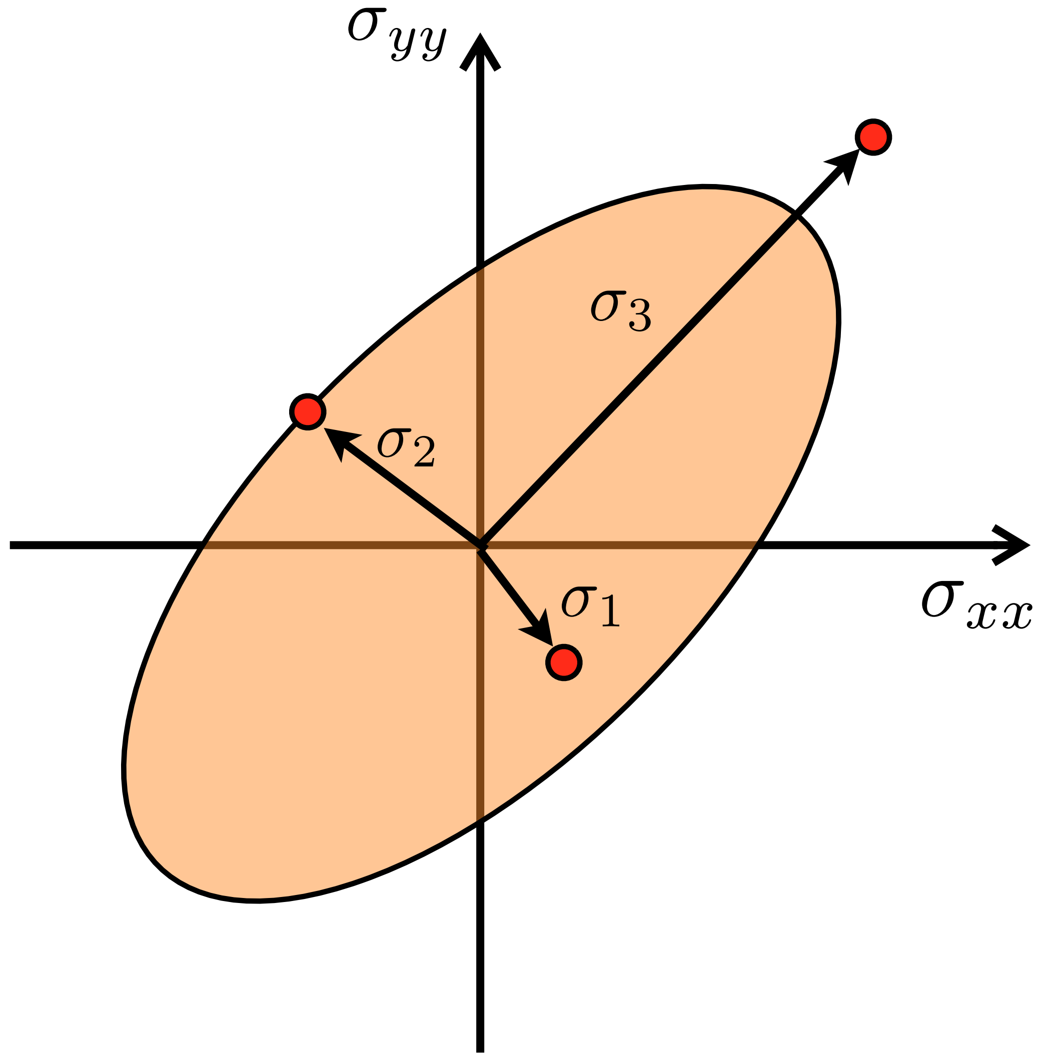
$$f = |\sigma| - \sigma_Y$$

$$f(\boldsymbol{\sigma}) = \quad f = \varphi(\boldsymbol{\sigma}) - \sigma_Y$$

Yield criterion

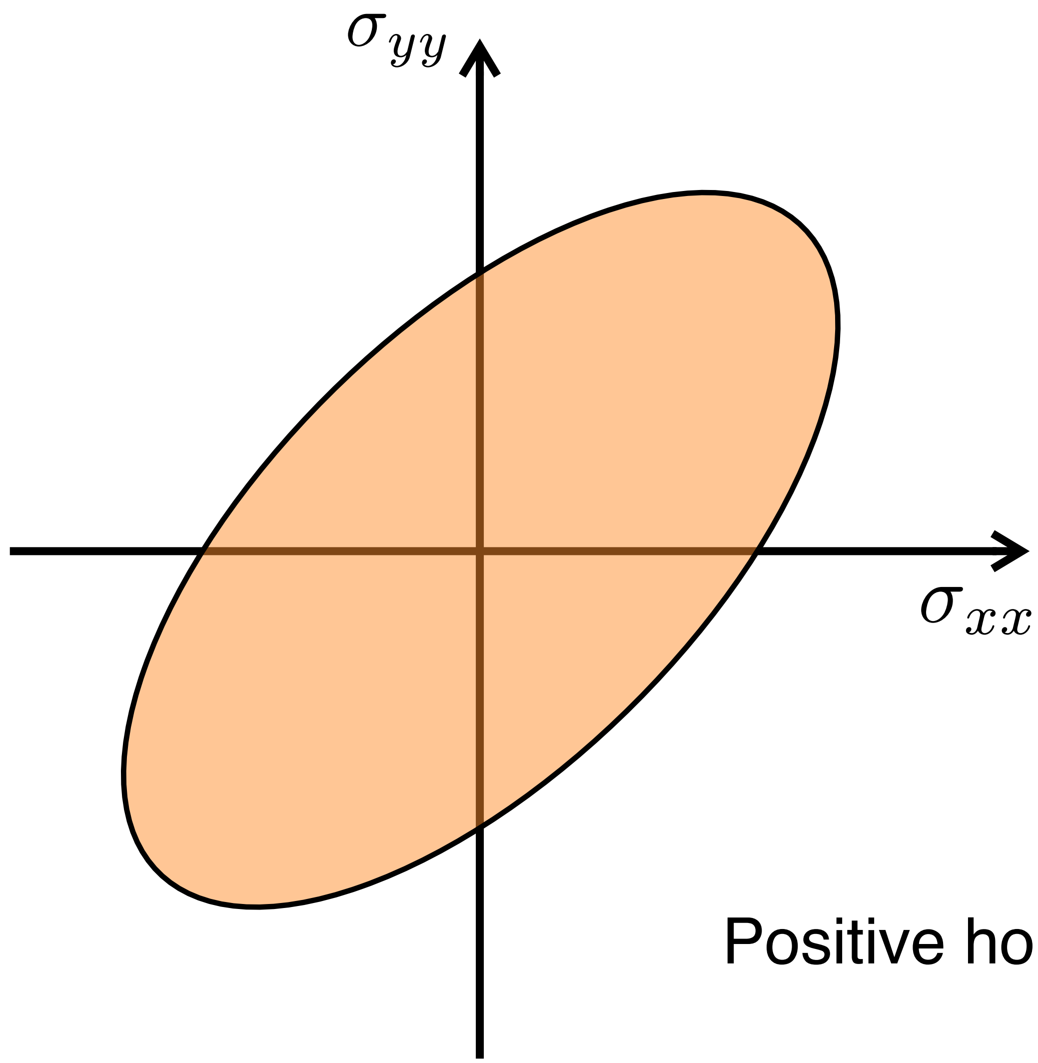
Equivalent stress

Yield criterion



Elastic domain	Yield criterion	Inadmissible region
$f(\boldsymbol{\sigma}) < 0$	$f(\boldsymbol{\sigma}) = 0$	$f(\boldsymbol{\sigma}) > 0$

Yield criterion



Positive homogeneous function of order one:

$$\varphi(a\boldsymbol{\sigma}) = a\varphi(\boldsymbol{\sigma}), a \geq 0$$

Euler's theorem for homogeneous functions:

$$\sigma_{ij}\frac{\partial\varphi(\boldsymbol{\sigma})}{\partial\sigma_{ij}} = \varphi(\boldsymbol{\sigma})$$

Assumed form:

$$f(\boldsymbol{\sigma}) = \varphi(\boldsymbol{\sigma}) - \sigma_Y$$

$$\sigma_{eq} = \varphi(\boldsymbol{\sigma})$$

Yield criterion

$$f(\boldsymbol{\sigma}) = \varphi(\boldsymbol{\sigma}) - \sigma_Y \qquad \sigma_{eq} = \varphi(\boldsymbol{\sigma})$$

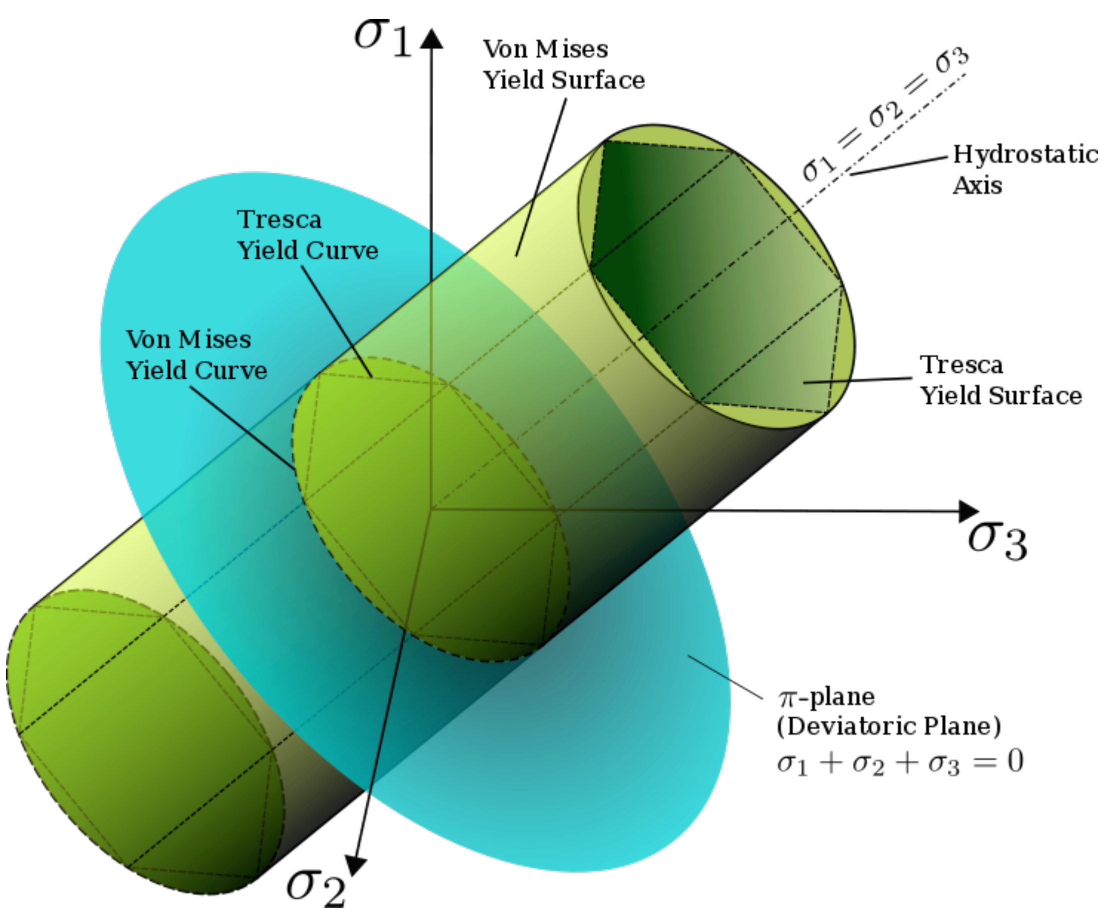
?

von Mises criterion:

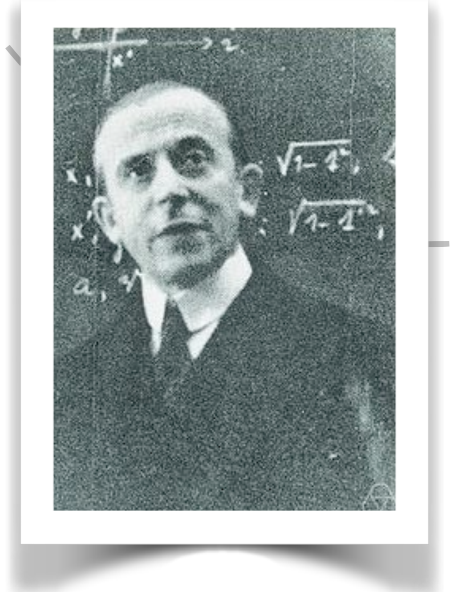
$$f(\boldsymbol{\sigma}) = \sqrt{3J_2} - \sigma_Y$$

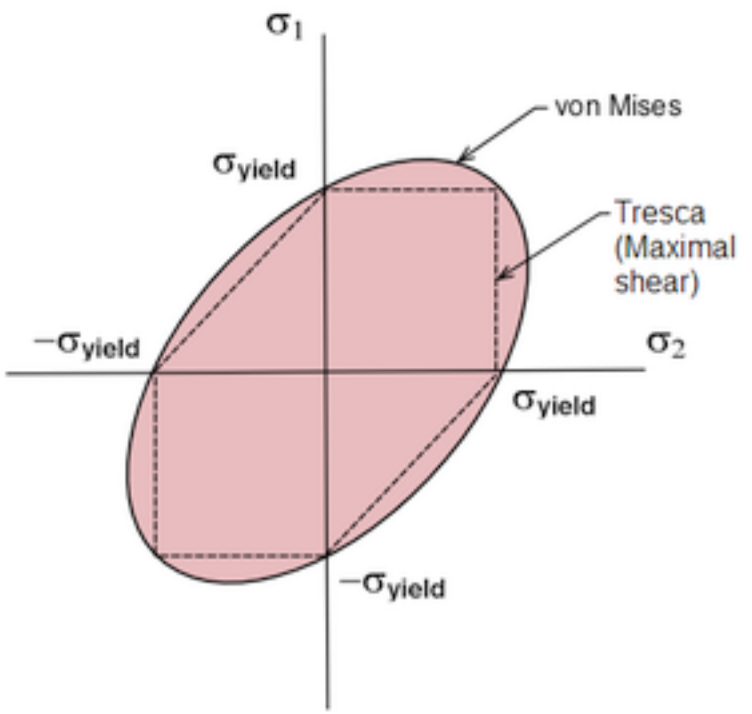
$$f(\boldsymbol{\sigma}) = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} - \sigma_Y$$

Valid for:
- isotropic materials
- pressure insensitive



von Mises criterion 1913





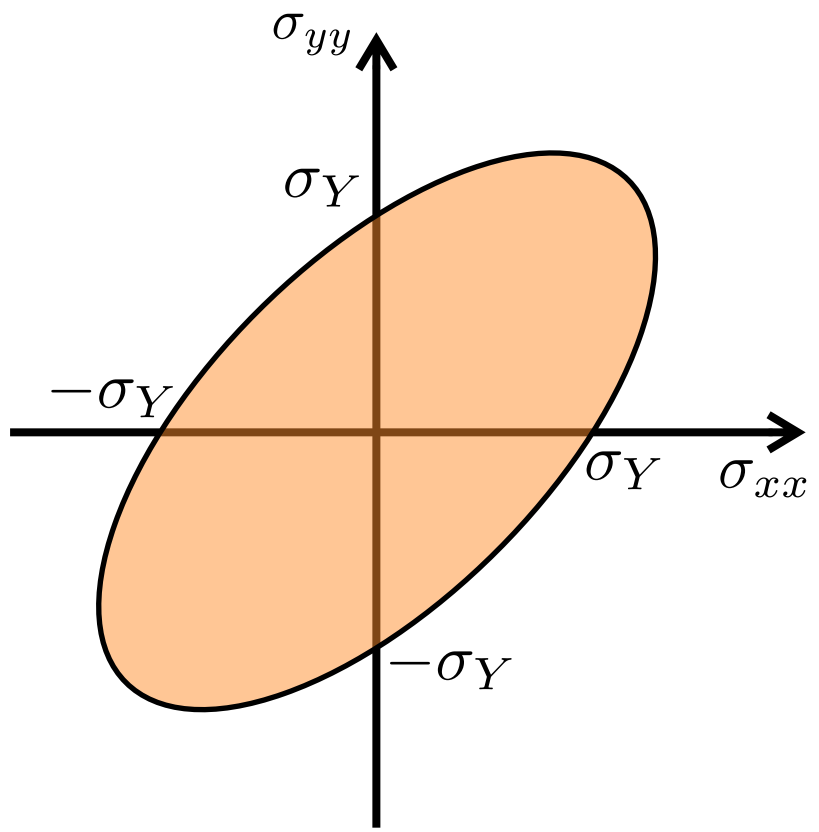
Yield criterion

$$f(\boldsymbol{\sigma}) = \varphi(\boldsymbol{\sigma}) - \sigma_Y \qquad \sigma_{eq} = \varphi(\boldsymbol{\sigma})$$

$$f(\boldsymbol{\sigma}) = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} - \sigma_Y$$

Anisotropic

Pressure sensitivity

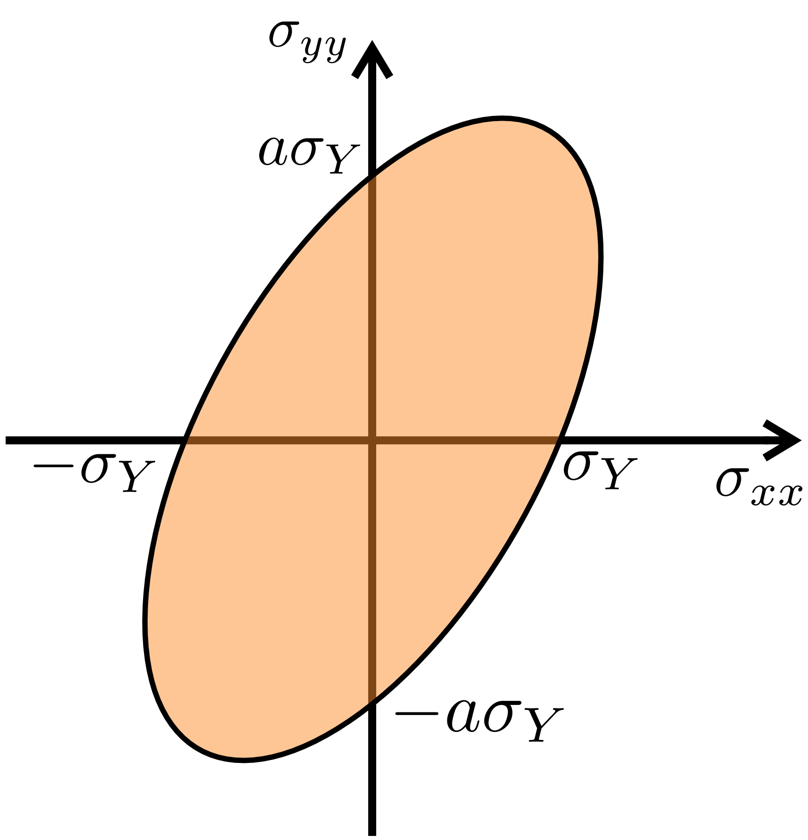


Yield criterion

$$f(\boldsymbol{\sigma}) = \varphi(\boldsymbol{\sigma}) - \sigma_Y \qquad \sigma_{eq} = \varphi(\boldsymbol{\sigma})$$

↓

Anisotropic



Rodney Hill
1948

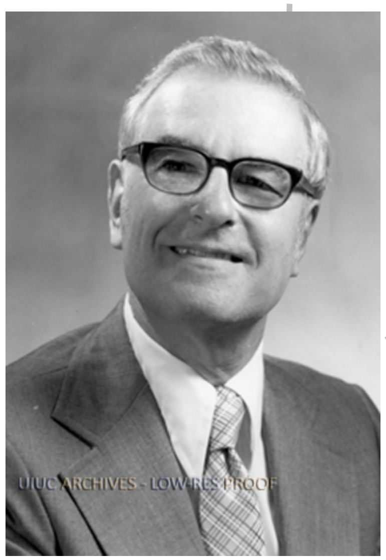
Hill criterion:

$$\varphi^2(\boldsymbol{\sigma}) = F(\sigma_{22} - \sigma_{33})^2 + G(\sigma_{33} - \sigma_{11})^2 + H(\sigma_{11} - \sigma_{22})^2$$
$$+ \; L(\sigma_{23}^2 - \sigma_{32}^2) + M(\sigma_{13}^2 - \sigma_{31}^2) + N(\sigma_{12}^2 - \sigma_{21}^2)$$

Yield criterion

$$f(\boldsymbol{\sigma}) = \varphi(\boldsymbol{\sigma}) - \sigma_Y \qquad \sigma_{eq} = \varphi(\boldsymbol{\sigma})$$

Pressure dependency

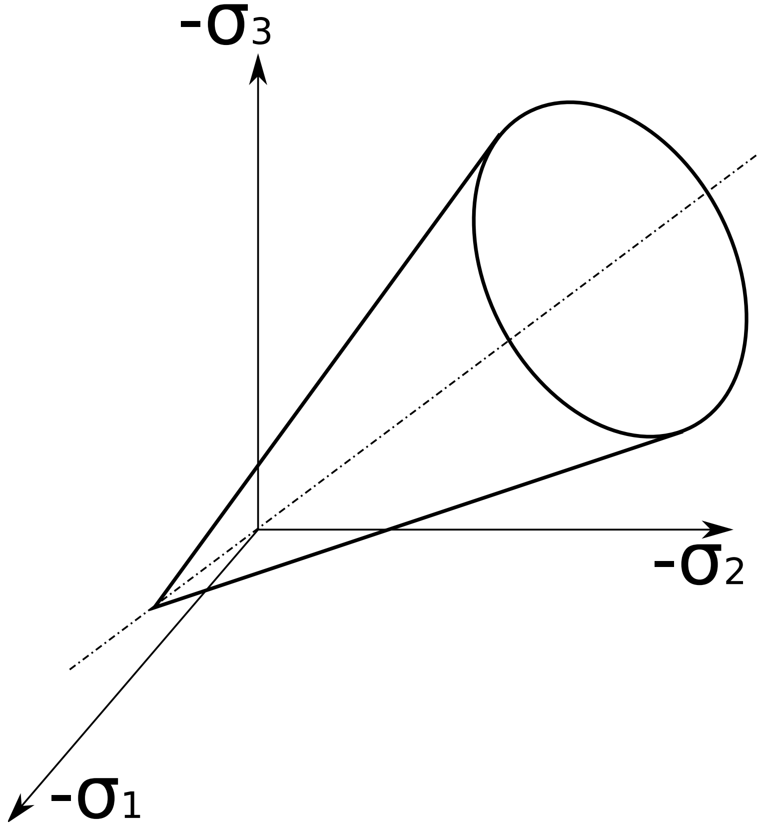


Drucker

Prager

Drucker-Prager criterion 1952
:

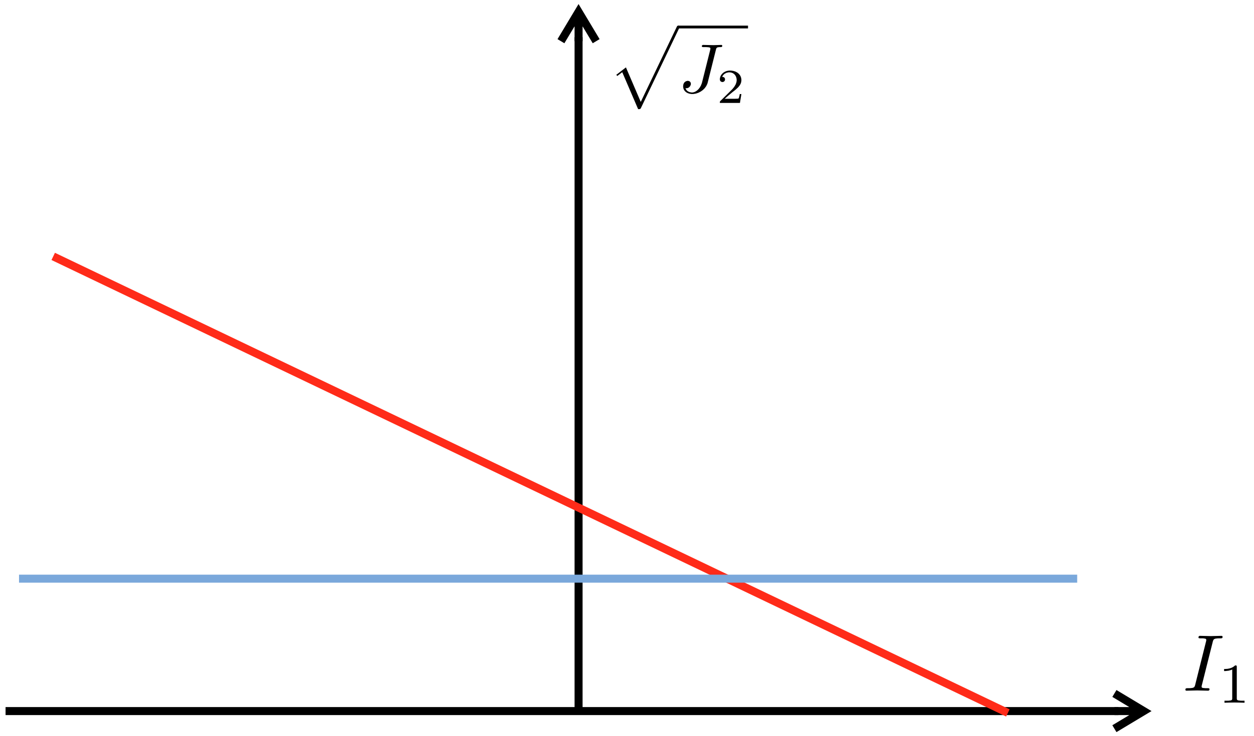
$$\varphi(\boldsymbol{\sigma}) = \sqrt{J_2} - \alpha I_1$$



Yield criterion

$$f(\boldsymbol{\sigma}) = \varphi(\boldsymbol{\sigma}) - \sigma_Y \qquad \sigma_{eq} = \varphi(\boldsymbol{\sigma})$$

$\sqrt{J_2}, I_1$ plane



Yield criterion

$$f(\boldsymbol{\sigma}) = \varphi(\boldsymbol{\sigma}) - \sigma_Y \qquad \sigma_{eq} = \varphi(\boldsymbol{\sigma})$$

↓

Can be “customized” !

$$f(\boldsymbol{\sigma}) = f(\sigma_1, \sigma_2, \sigma_3) = 0 \qquad f(\boldsymbol{\sigma}) = f(I_\sigma, II_\sigma, III_\sigma) = 0$$

For isotropic materials:

$$f(\boldsymbol{\sigma}) = f(J_2, J_3) = 0$$

$$f(\boldsymbol{\sigma}) = f(I_1, J_2, J_3) = 0$$

Yield criterion

$$f(\boldsymbol{\sigma}) = \varphi(\boldsymbol{\sigma}) - \sigma_Y \qquad \sigma_{eq} = \varphi(\boldsymbol{\sigma})$$

Can be “customized" !

The Deshpande-Fleck-Ashby yield criterion:

$$\begin{aligned}\varphi^2(\boldsymbol{\sigma}) = & F(\sigma_{22} - \sigma_{33})^2 + G(\sigma_{33} - \sigma_{11})^2 + H(\sigma_{11} - \sigma_{22})^2 \\ & + L(\sigma_{23}^2 - \sigma_{32}^2) + M(\sigma_{13}^2 - \sigma_{31}^2) + N(\sigma_{12}^2 - \sigma_{21}^2) \\ & + K(\sigma_{11} + \sigma_{22} + \sigma_{33})\end{aligned}$$

Anisotropic + Isotropic pressure sensitivity

Yield criterion

$$f(\boldsymbol{\sigma}) = \varphi(\boldsymbol{\sigma}) - \sigma_Y \qquad \sigma_{eq} = \varphi(\boldsymbol{\sigma})$$

↓

Can be "customized" !

The Caddell-Raghava-Atkins yield criterion:

$$\begin{aligned}\varphi^2(\boldsymbol{\sigma}) = {} & F(\sigma_{22}-\sigma_{33})^2 + G(\sigma_{33}-\sigma_{11})^2 + H(\sigma_{11}-\sigma_{22})^2 \\ & + L(\sigma_{23}^2-\sigma_{32}^2) + M(\sigma_{13}^2-\sigma_{31}^2) + N(\sigma_{12}^2-\sigma_{21}^2) \\ & + I\sigma_{11} + J\sigma_{22} + K\sigma_{33}\end{aligned}$$

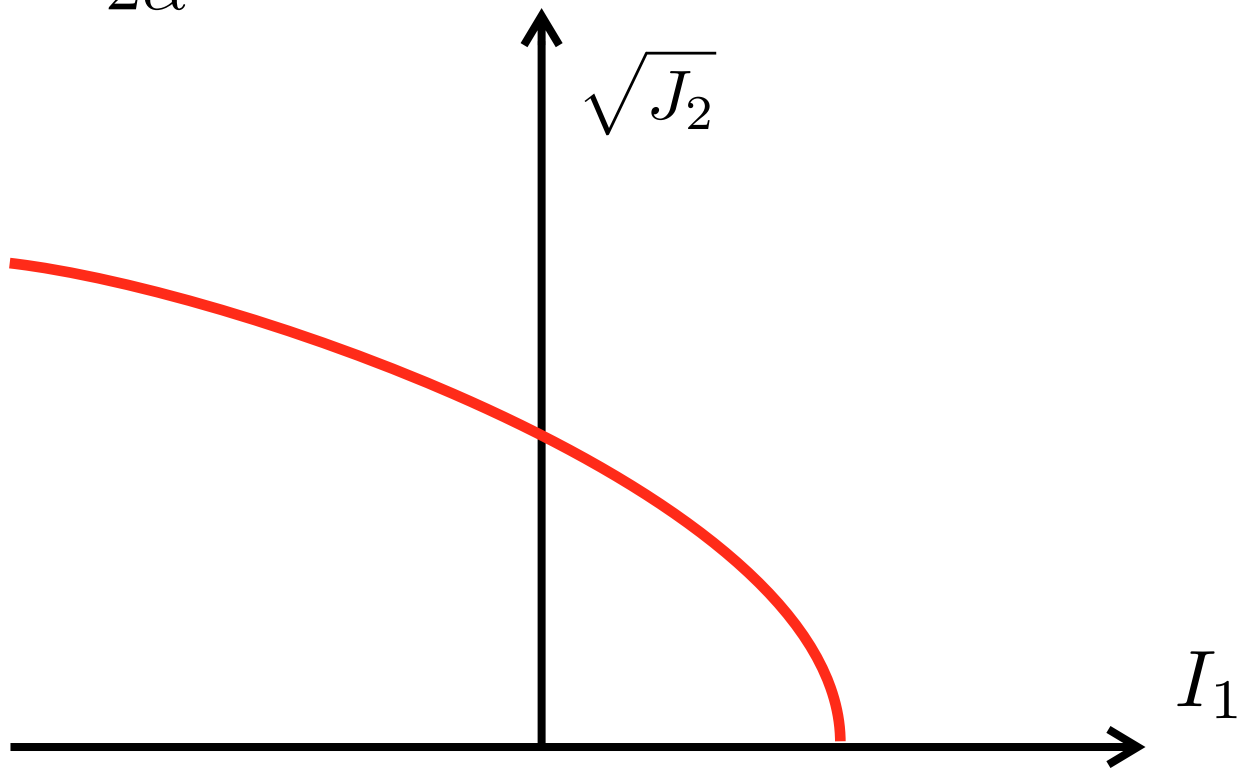
Anisotropic + anisotropic pressure sensitivity

Yield criterion

$$f(\boldsymbol{\sigma}) = \varphi(\boldsymbol{\sigma}) - \sigma_Y \qquad \sigma_{eq} = \varphi(\boldsymbol{\sigma})$$

The Raghava yield criterion:

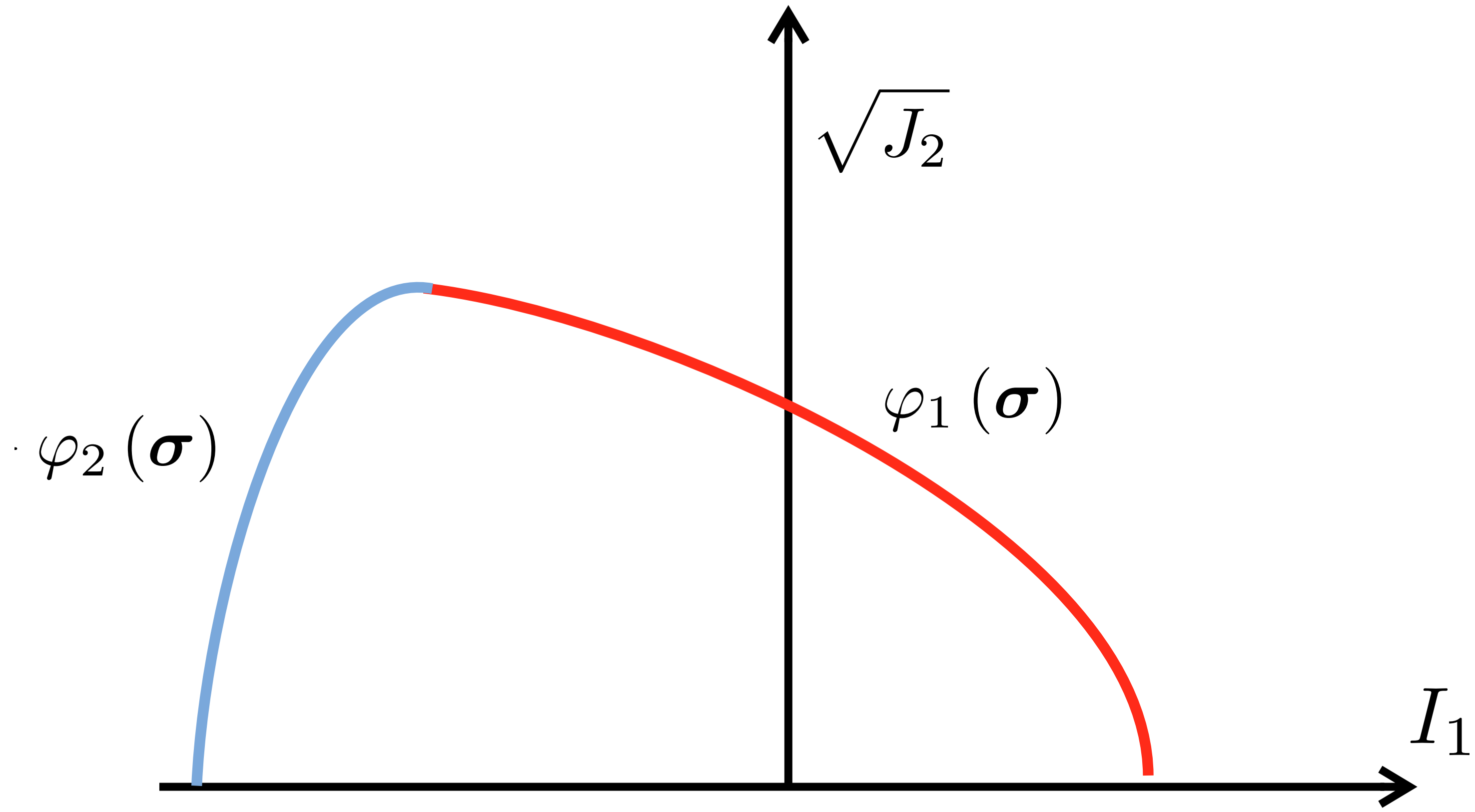
$$\varphi(\boldsymbol{\sigma}) = \frac{(\alpha - 1) I_1 + \sqrt{(\alpha - 1)^2 I_1^2 + 12\alpha J_2}}{2\alpha}$$



Yield criterion

$$f(\boldsymbol{\sigma}) = \varphi(\boldsymbol{\sigma}) - \sigma_Y \qquad \sigma_{eq} = \varphi(\boldsymbol{\sigma})$$

$$\varphi(\boldsymbol{\sigma}) = \varphi_1(\boldsymbol{\sigma}) + \varphi_2(\boldsymbol{\sigma})$$

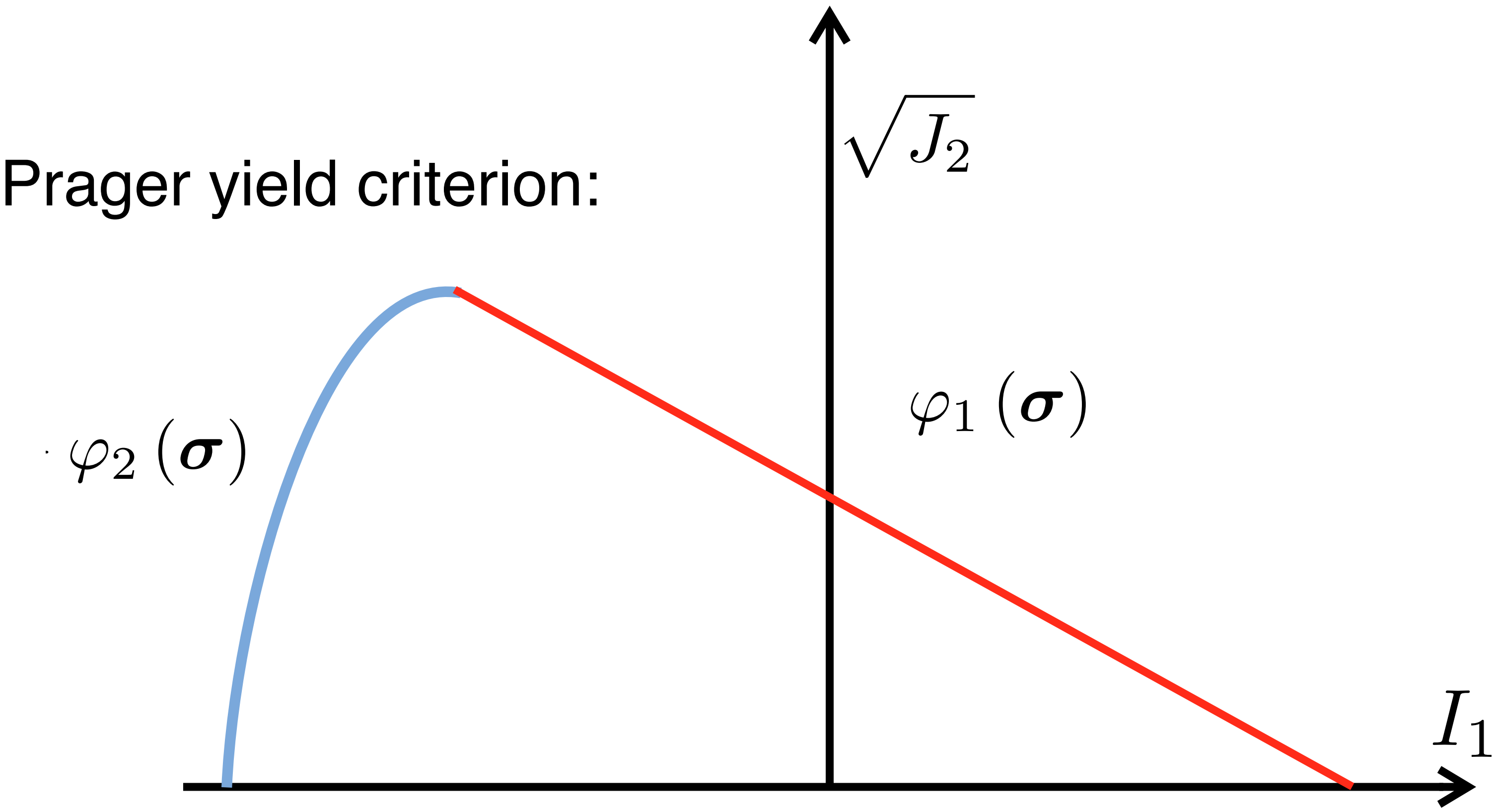


Yield criterion

$$f(\boldsymbol{\sigma}) = \varphi(\boldsymbol{\sigma}) - \sigma_Y \qquad \sigma_{eq} = \varphi(\boldsymbol{\sigma})$$

$$\varphi(\boldsymbol{\sigma}) = \varphi_1(\boldsymbol{\sigma}) + \varphi_2(\boldsymbol{\sigma})$$

Capped Drucker-Prager yield criterion:



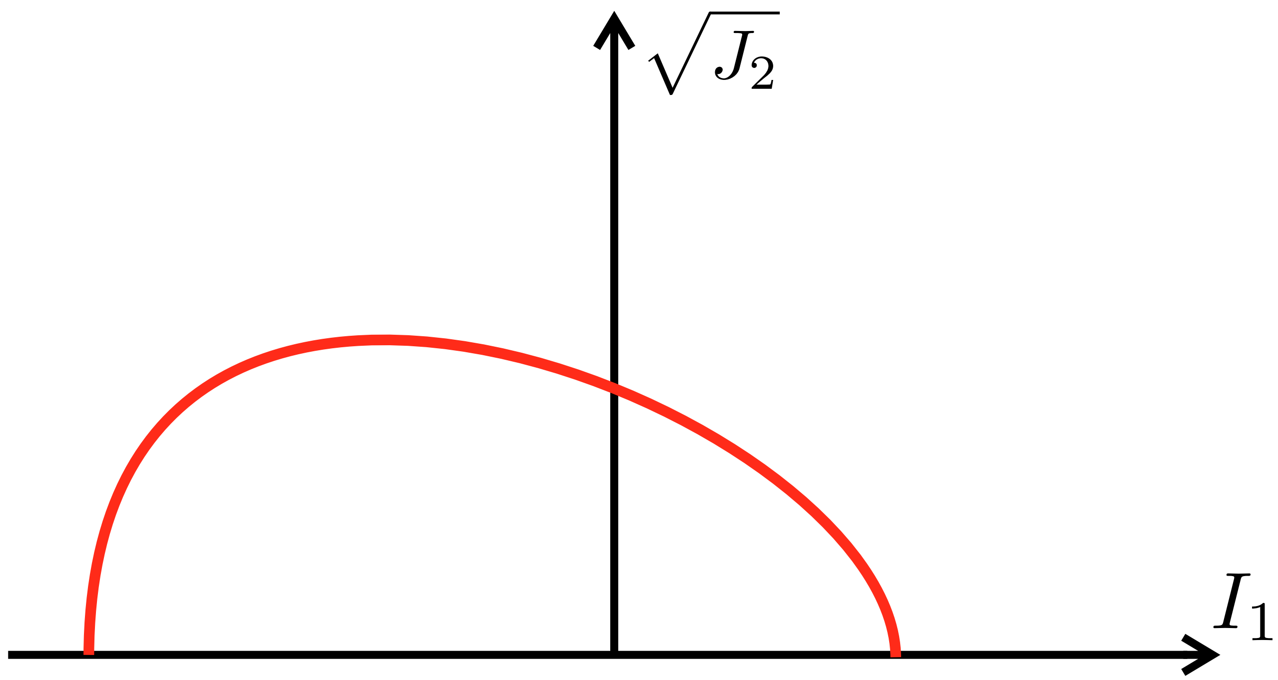
$$\varphi_2(\boldsymbol{\sigma}) = \frac{1}{\beta^2}(I_1 - p_t + a)^2 + \left(\frac{\sqrt{J_2}}{M}\right)^2 - a^2 \qquad \varphi_1(\boldsymbol{\sigma}) = \sqrt{J_2} - \alpha I_1$$

Yield criterion

$$f(\boldsymbol{\sigma}) = \varphi(\boldsymbol{\sigma}) - \sigma_Y \qquad \sigma_{eq} = \varphi(\boldsymbol{\sigma})$$

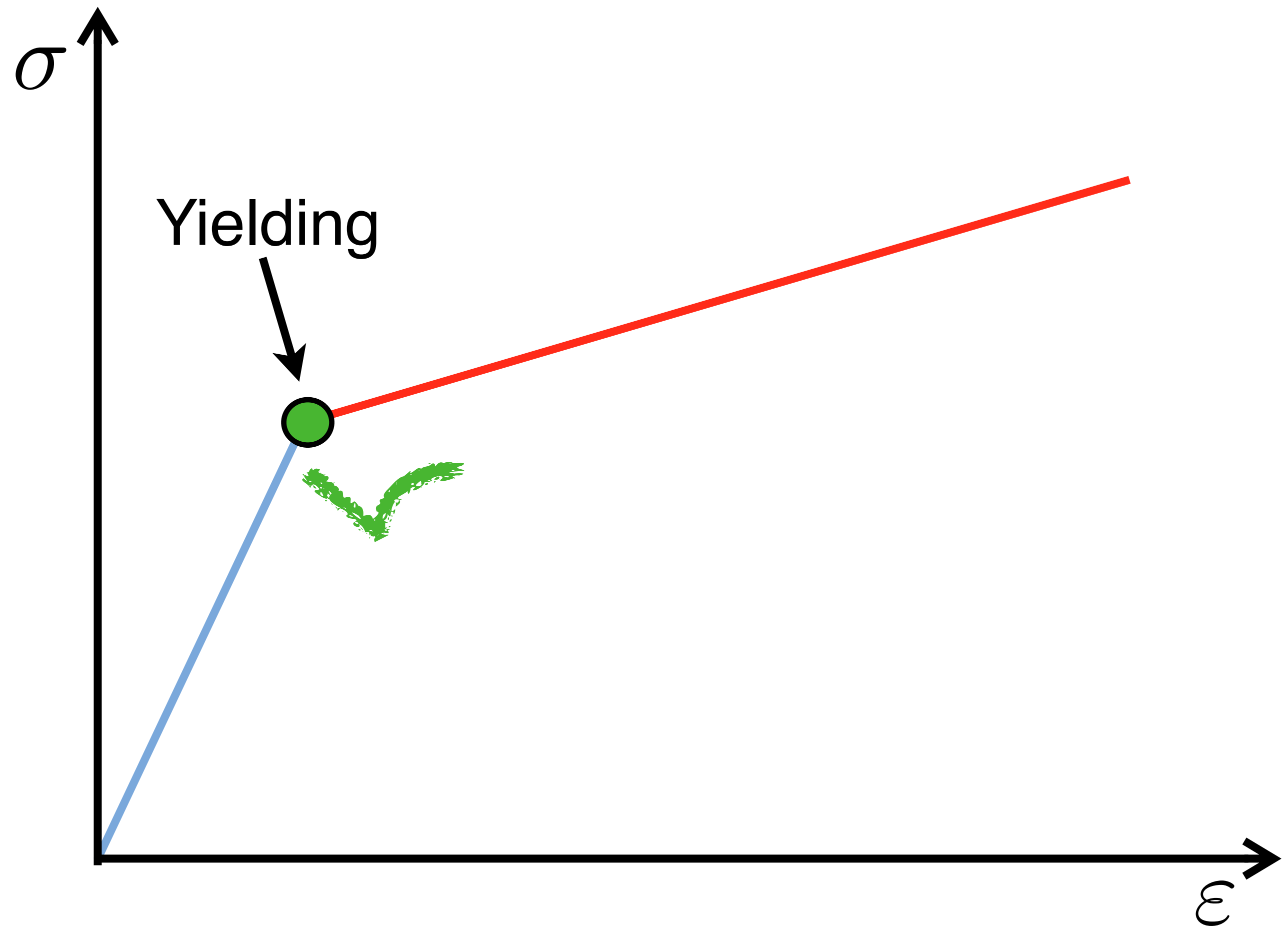
Clam-Clay yield criterion:

$$\varphi(\boldsymbol{\sigma}) = \frac{1}{\beta^2}\left(I_1 - p_t + a\right)^2 + \left(\frac{\sqrt{J_2}}{M}\right)^2 - a^2$$



Yield criterion

Assumed form: $f\left(\boldsymbol{\sigma}\right)=\varphi\left(\boldsymbol{\sigma}\right)-\sigma_Y$

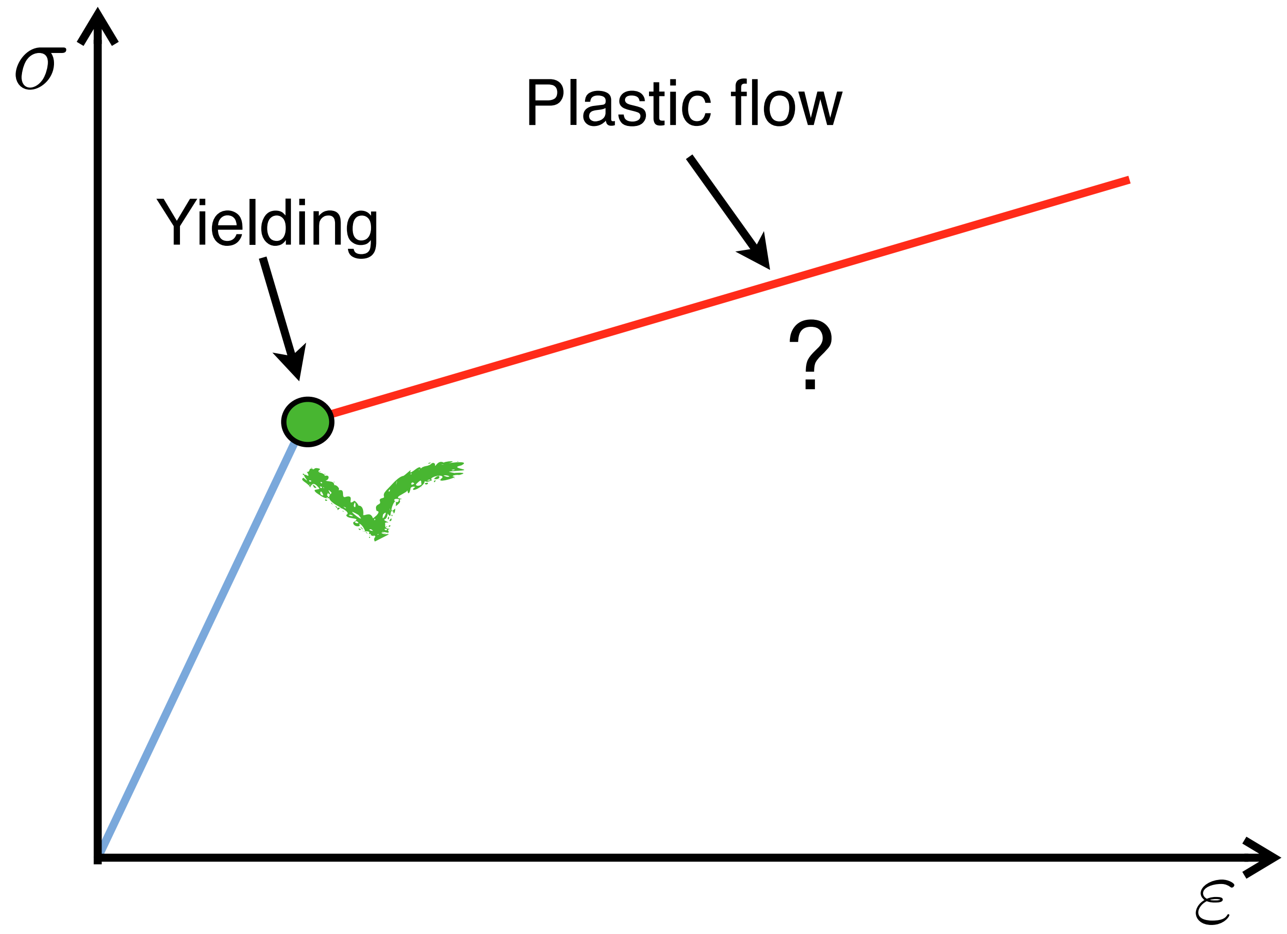


$$\varphi\left(a\boldsymbol{\sigma}\right)=a\varphi\left(\boldsymbol{\sigma}\right),a\geq 0$$

$$\sigma_{ij}\frac{\partial\varphi\left(\boldsymbol{\sigma}\right)}{\partial\sigma_{ij}}=\varphi\left(\boldsymbol{\sigma}\right)$$

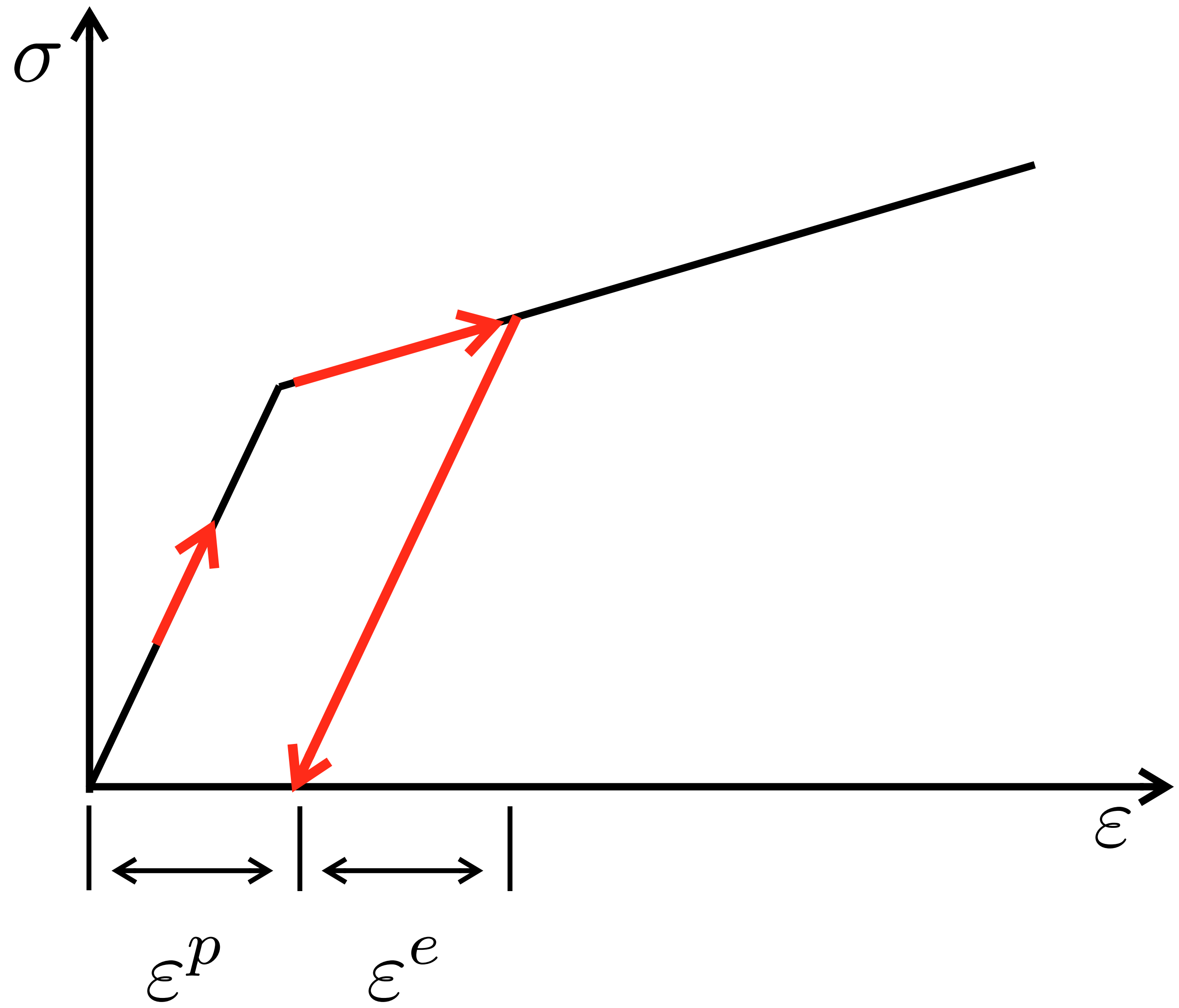
$$f\left(\boldsymbol{\sigma}\right)=f\left(\sigma_1,\sigma_2,\sigma_3\right)=0$$

Flow rule



Flow rule

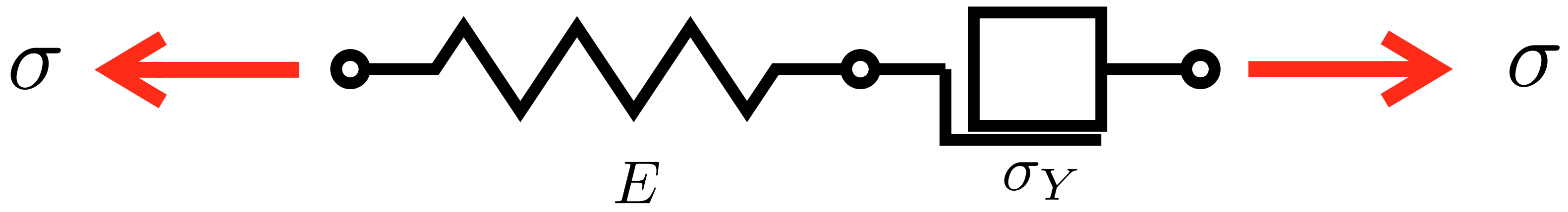
$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p}$$



We need an evolution rule for ε^p

Flow rule

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p} \qquad \dot{\varepsilon}^{p} = ?$$



$$f = |\sigma| - \sigma_Y$$

$$\sigma > 0 \longrightarrow \dot{\varepsilon}^{p} > 0$$

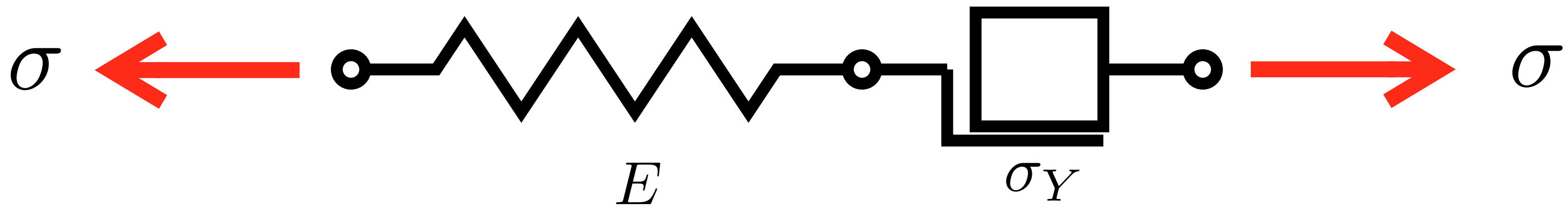
$$\mathfrak{D} = \sigma_{ij}\dot{\varepsilon}_{ij}^{p} \geq 0$$

$$\sigma < 0 \longrightarrow \dot{\varepsilon}^{p} < 0$$

$$\dot{\varepsilon}^{p} = \dot{\lambda} sgn(\sigma) \qquad \dot{\lambda} \geq 0$$

Flow rule

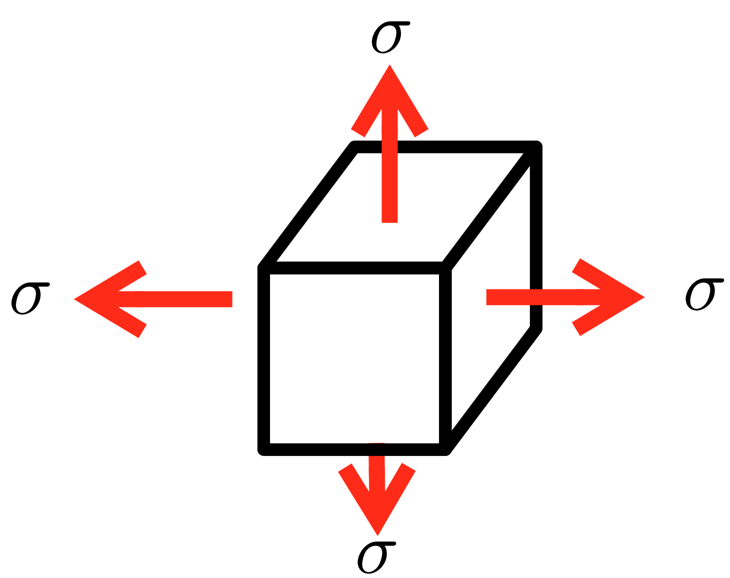
$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p} \qquad \dot{\varepsilon}^{p} = ?$$



$$f = |\sigma| - \sigma_Y$$

$$\dot{\varepsilon}^{p} = \dot{\lambda}\, sign(\sigma)$$

$$\dot{\varepsilon}^{p} = \dot{\lambda}\frac{\partial f}{\partial \sigma}$$



$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

$$\dot{\boldsymbol{\varepsilon}}^{p} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\frac{\partial \varphi(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \qquad \dot{\lambda} \geq 0$$

→ Flow vector

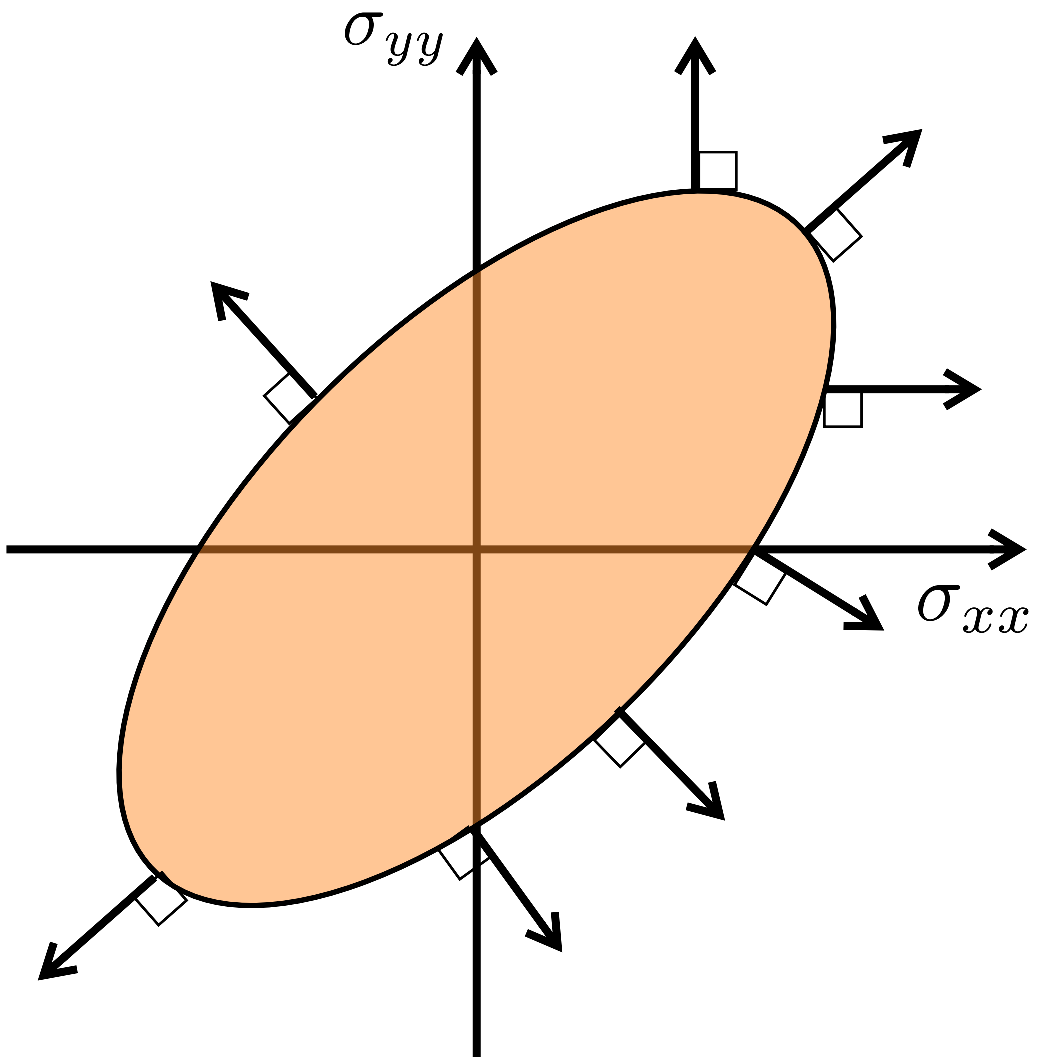
Flow rule

Plane stress visualization of a plastic flow

von Mises criterion:

$$f(\boldsymbol{\sigma}) = \sqrt{3J_2} - \sigma_Y$$

$$\frac{\partial \varphi(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = \frac{3\dot{\lambda}}{2\varphi}\boldsymbol{\sigma}'$$



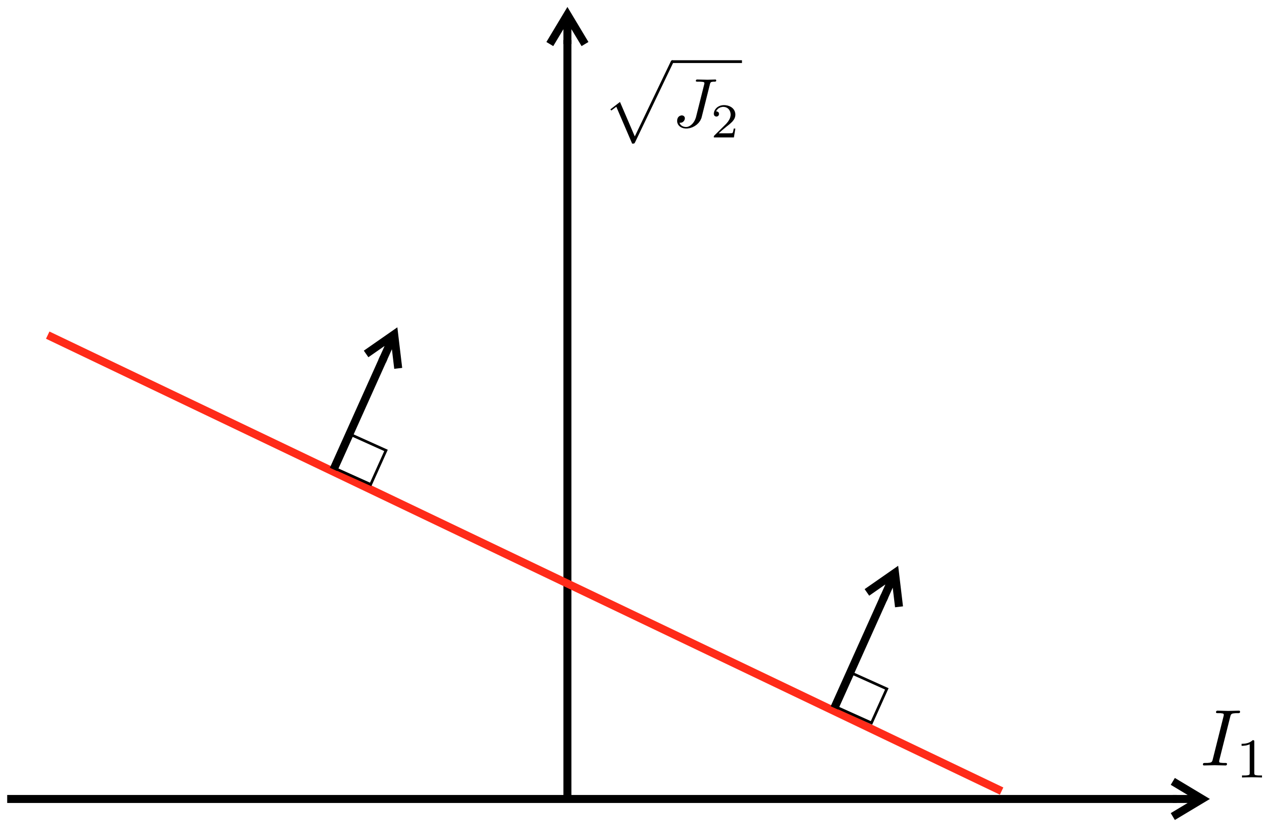
Flow rule

$\sqrt{J_2}, I_1$ Plane visualization of a plastic flow

Drucker-Prager criterion :

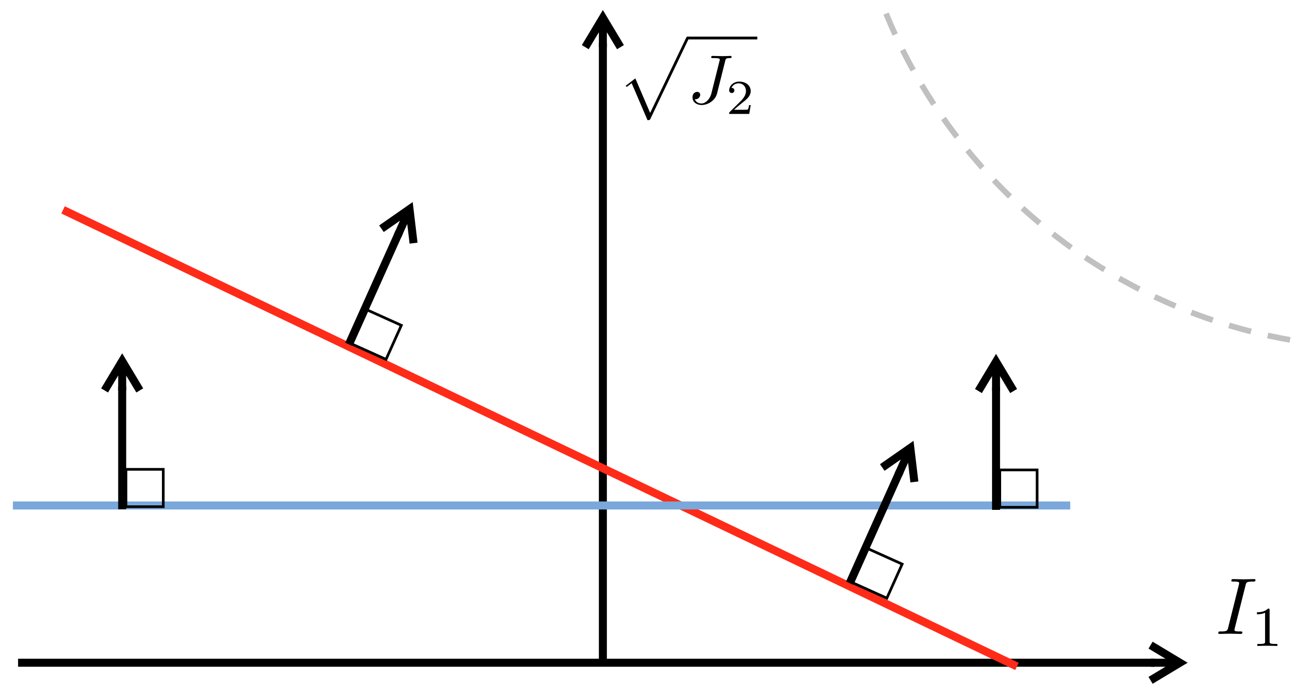
$$\varphi\left(\boldsymbol{\sigma}\right) = \sqrt{J_2} - \alpha I_1$$

$$\frac{\partial \varphi\left(\boldsymbol{\sigma}\right)}{\partial \boldsymbol{\sigma}} = \frac{\dot{\lambda}}{2\sqrt{J_2}}\boldsymbol{\sigma}^{'} + \alpha \mathbf{I}$$



Flow rule

$\sqrt{J_2}, I_1$ Plane visualization of a plastic flow



Drucker-Prager criterion :

$$\varphi(\boldsymbol{\sigma}) = \sqrt{J_2} - \alpha I_1$$

$$\frac{\partial \varphi(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = \frac{\dot{\lambda}}{2\sqrt{J_2}}\boldsymbol{\sigma}' + \alpha \mathbf{I}$$

von Mises criterion:

$$f(\boldsymbol{\sigma}) = \sqrt{3J_2} - \sigma_Y$$

$$\frac{\partial \varphi(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = \frac{3\dot{\lambda}}{2\varphi}\boldsymbol{\sigma}'$$

vM:

$$trace\left(\frac{\partial \varphi(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}\right) = 0$$

$$trace(\boldsymbol{\varepsilon}^p) = \varepsilon_V^p$$

DP:

$$trace\left(\frac{\partial \varphi(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}\right) = 3\alpha$$

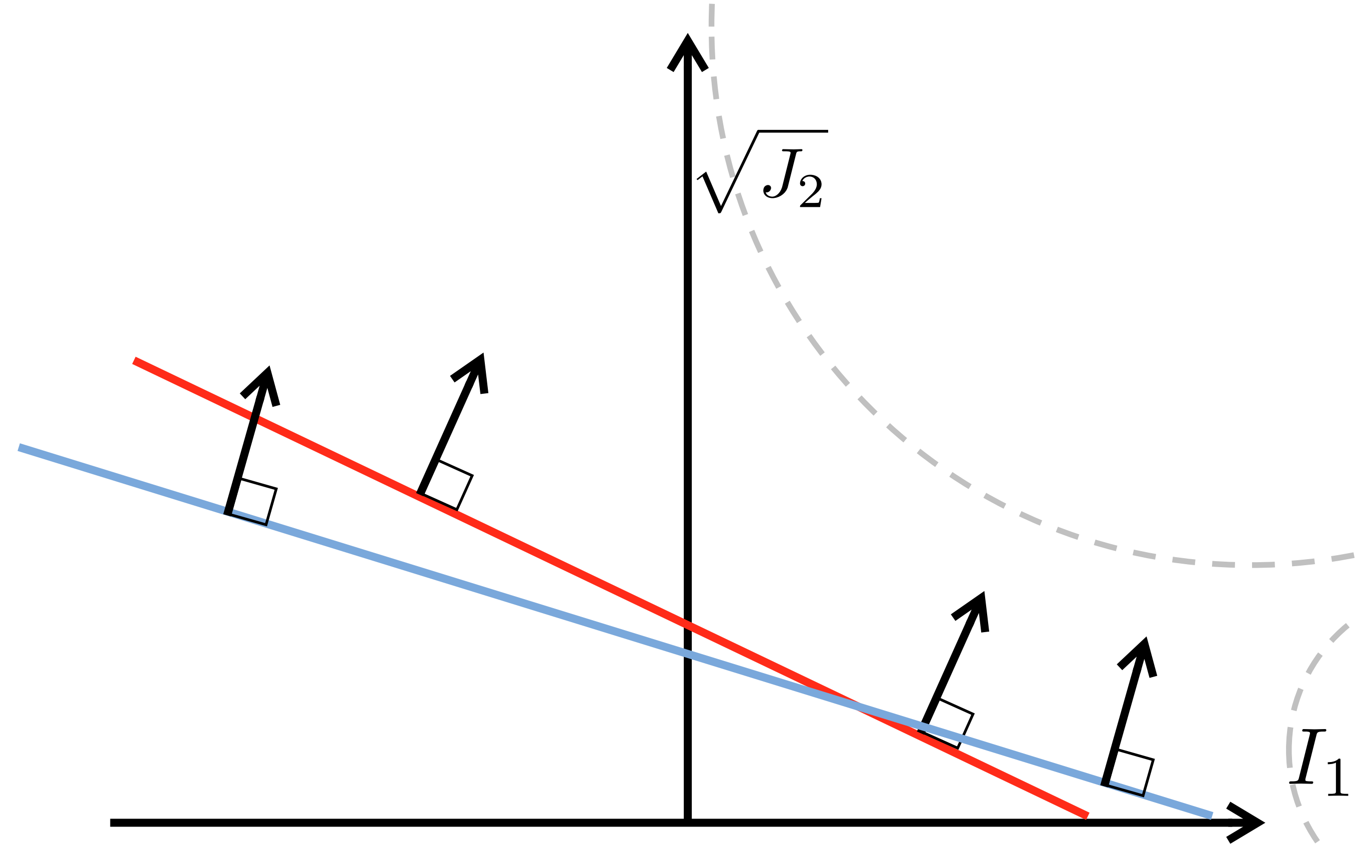
Flow rule

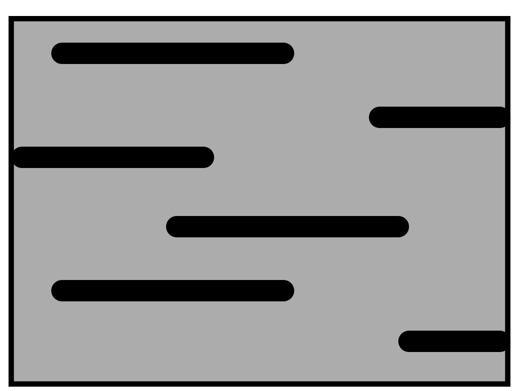
Associated plasticity:

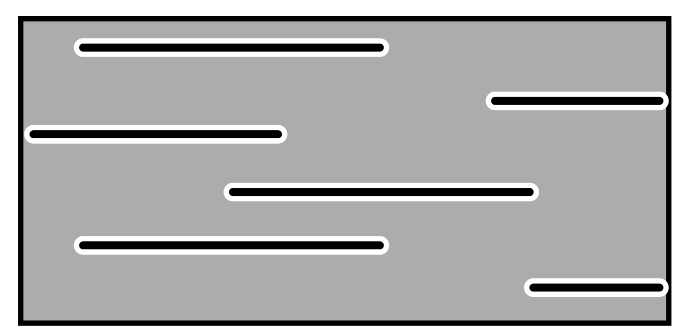
$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\frac{\partial \varphi\left(\boldsymbol{\sigma}\right)}{\partial \boldsymbol{\sigma}}$$

Non-associated plasticity:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial g}{\partial \boldsymbol{\sigma}}$$







Flow rule

Potential restrictions on the flow potential functions:

Positive homogeneous function of order one:

$$g = g\left(a\boldsymbol{\sigma}\right) > 0 \qquad g\left(a\boldsymbol{\sigma}\right) = a g\left(\boldsymbol{\sigma}\right), a \geq 0$$

Euler's theorem for homogeneous functions:

$$\sigma_{ij}\frac{\partial g\left(\boldsymbol{\sigma}\right)}{\partial \sigma_{ij}} = g\left(\boldsymbol{\sigma}\right)$$

Flow rule

Plastic dissipation has to be positive:

$$\mathfrak{D} = \sigma_{ij}\dot{\varepsilon}_{ij}^{p} \geq 0$$
$$= \sigma_{ij}\dot{\lambda}\frac{\partial g\left(\boldsymbol{\sigma}\right)}{\partial\sigma_{ij}}$$
$$= \dot{\lambda}g\left(\boldsymbol{\sigma}\right)$$

$$\dot{\lambda} \geq 0$$

$$\mathfrak{D} = \sigma_{ij}\dot{\varepsilon}_{ij}^{p} \geq 0$$
$$= \sigma_{ij}\dot{\lambda}\frac{\partial f\left(\boldsymbol{\sigma}\right)}{\partial\sigma_{ij}}$$
$$= \dot{\lambda}f\left(\boldsymbol{\sigma}\right)$$

$$\dot{\lambda} \geq 0$$

Flow rule

Equivalent plastic strain: p

Associated plasticity

$$\sigma_{ij}\dot{\varepsilon}_{ij}^{p} = \sigma_{eq}\dot{p}$$

$$\sigma_{ij}\dot{\varepsilon}_{ij}^{p} = \sigma_{ij}\dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}$$

$$\sigma_{ij}\dot{\varepsilon}_{ij}^{p} = \sigma_{ij}\dot{\lambda}\frac{\partial \varphi}{\partial \sigma_{ij}}$$

$$\sigma_{ij}\dot{\varepsilon}_{ij}^{p} = \sigma_{eq}\dot{\lambda}$$

$$\sigma_{eq}\dot{\lambda} = \sigma_{eq}\dot{p}$$

$$p = \int_0^t \dot{p} = \int_0^t \dot{\lambda}$$

Flow rule

Equivalent plastic strain: p

Non-associated plasticity

$$\sigma_{ij}\dot{\varepsilon}^p_{ij} = \sigma_{eq}\dot{p}$$

$$\sigma_{ij}\dot{\varepsilon}^p_{ij} = \sigma_{ij}\dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}}$$

$$\sigma_{ij}\dot{\varepsilon}^p_{ij} = g\left(\sigma_{ij}\right)\dot{\lambda}$$

$$g\left(\sigma_{ij}\right)\dot{\lambda} = \sigma_{eq}\dot{p}$$

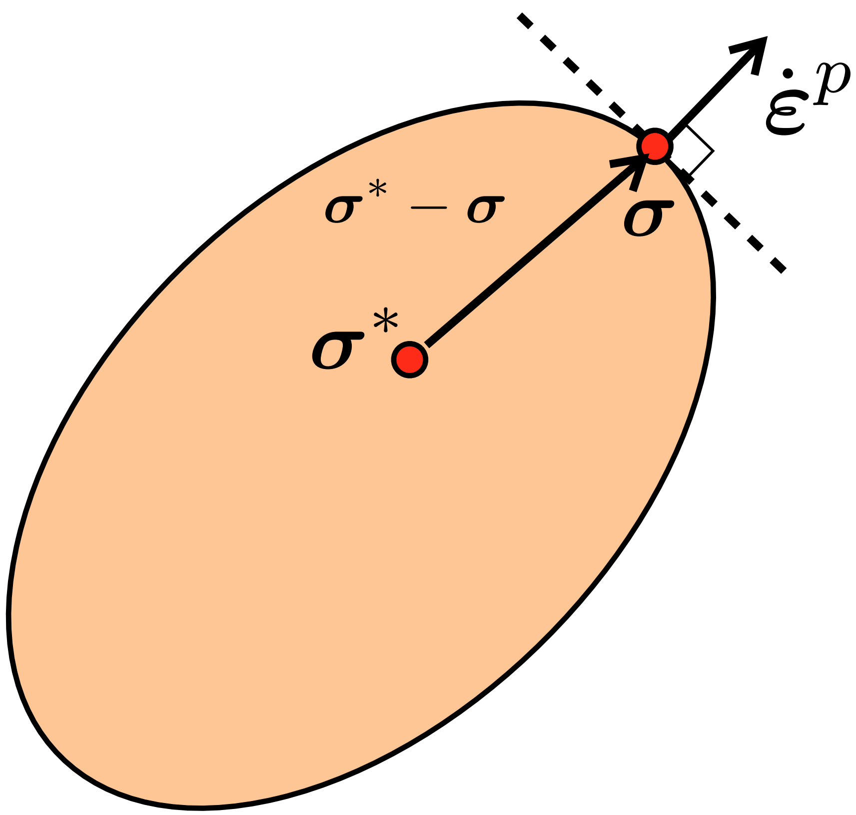
$$p = \int_0^t \dot{p} \neq \int_0^t \dot{\lambda}$$

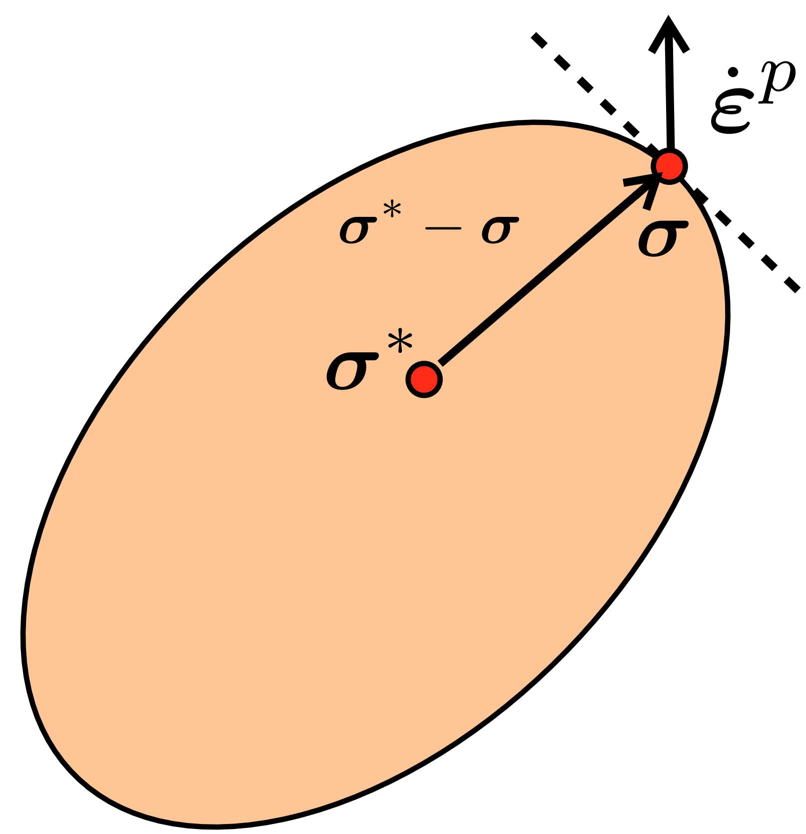
Flow rule

Principle of maximum plastic dissipation:

$$\left(\sigma_{ij} - \sigma_{ij}^{*}\right) \dot{\varepsilon}_{ij}^{p} \geq 0$$

$$\sigma_{ij} \dot{\varepsilon}_{ij}^{p} \geq \sigma_{ij}^{*} \dot{\varepsilon}_{ij}^{p}$$

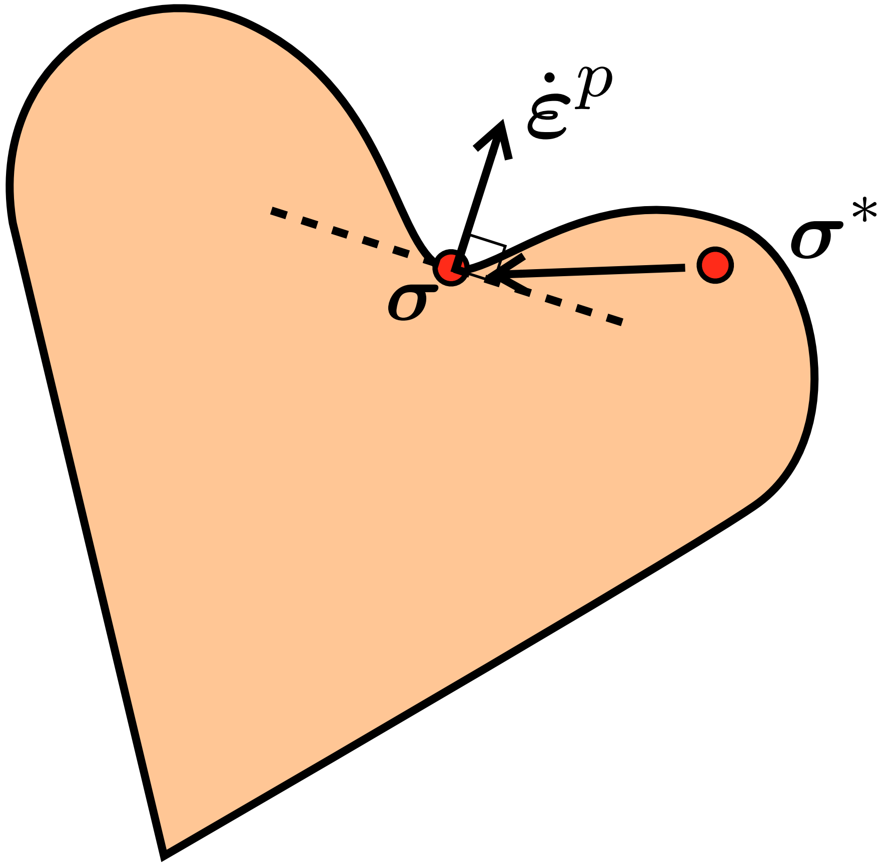




Flow rule

Principle of maximum plastic dissipation:

$$\left(\sigma_{ij} - \sigma_{ij}^{*}\right) \dot{\varepsilon}_{ij}^{p} \geq 0 \qquad \sigma_{ij}\dot{\varepsilon}_{ij}^{p} \geq \sigma_{ij}^{*}\dot{\varepsilon}_{ij}^{p}$$



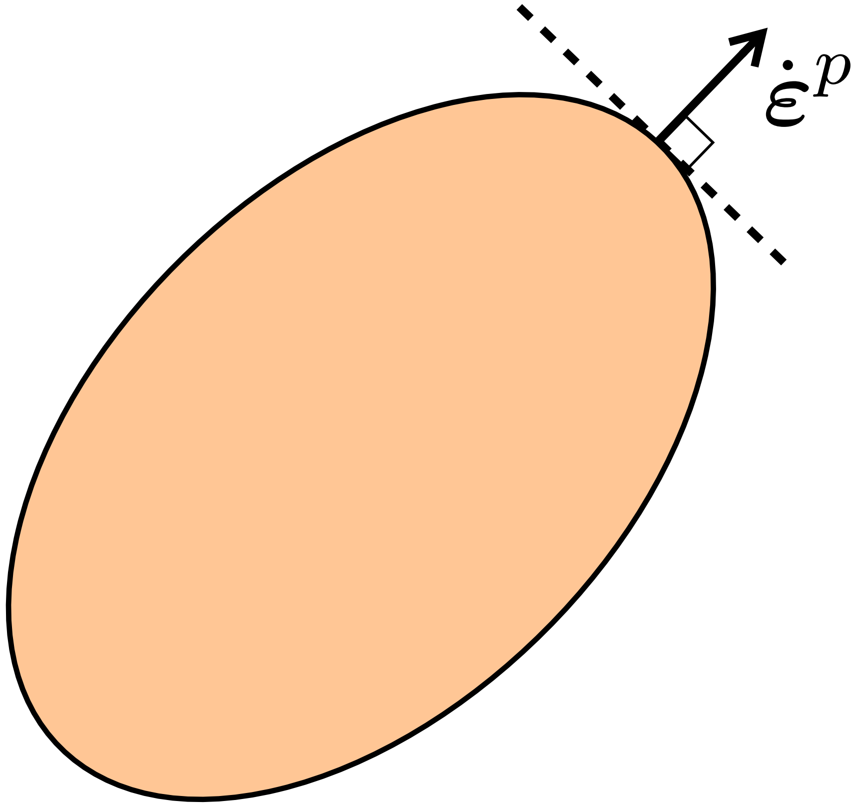
Flow rule

Principle of maximum plastic dissipation:

$$\left(\sigma_{ij} - \sigma^*_{ij}\right) \dot{\varepsilon}^p_{ij} \geq 0 \qquad \sigma_{ij}\dot{\varepsilon}^p_{ij} \geq \sigma^*_{ij}\dot{\varepsilon}^p_{ij}$$

implies:

- Convexity of the yield/flow potential
- Normality rule



Flow rule

Stress inside the elastic domain:

$$f < 0 \qquad \dot{\lambda} = 0$$

Stress on the yield surface:

$$f = 0 \qquad \dot{\lambda} \geq 0$$

$$\dot{\lambda} > 0$$

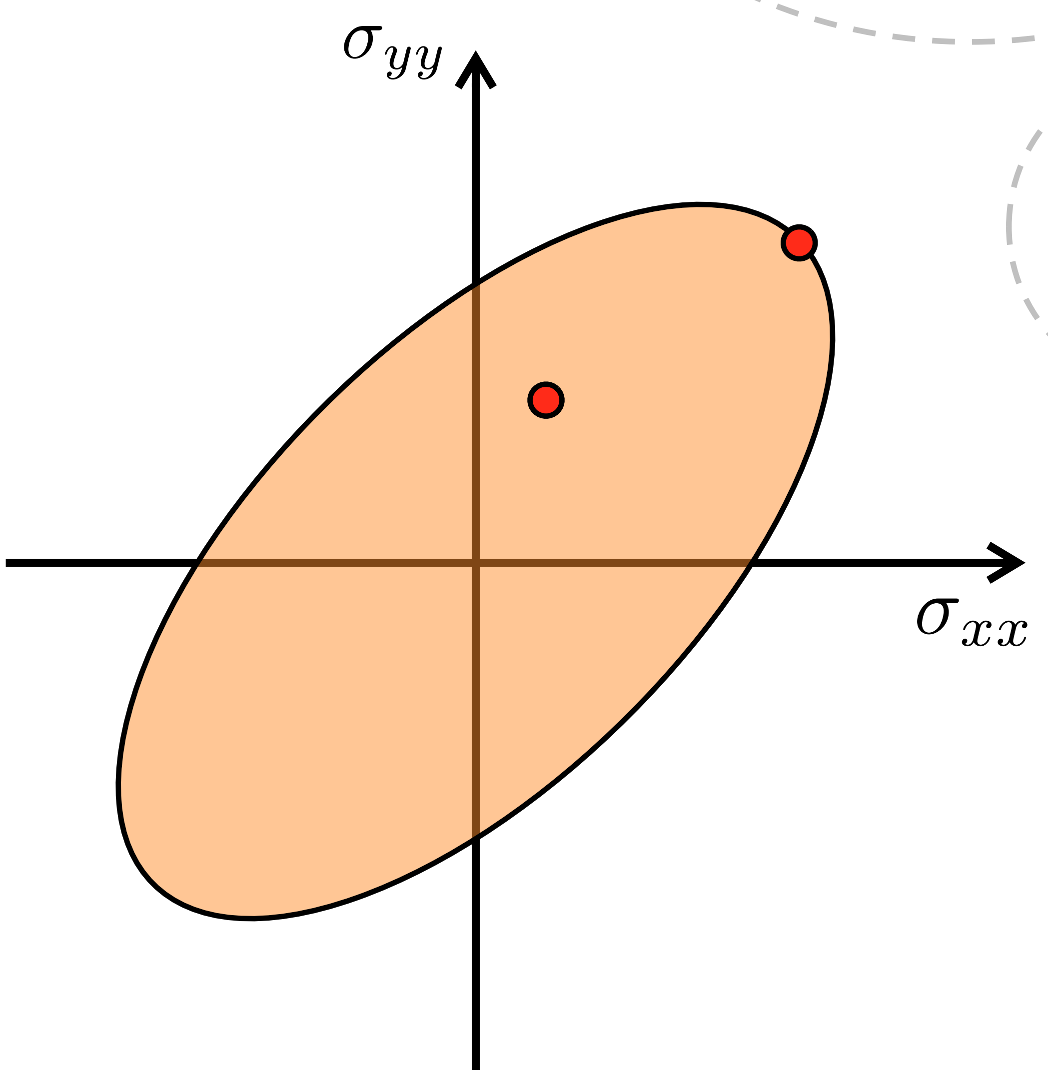
$$\dot{\lambda} = 0$$

Kuhn-Tucker conditions:

$$f \leq 0, \dot{\lambda} \geq 0, \dot{\lambda} f = 0$$

Consistency condition:

$$\dot{\lambda} \dot{f} = 0$$



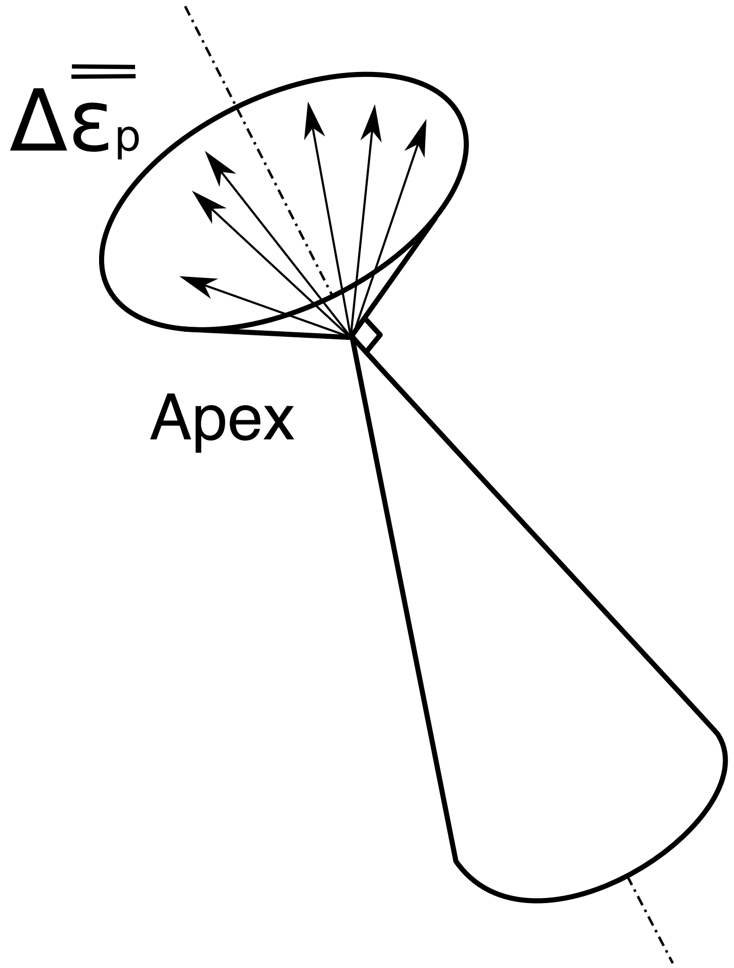
Flow rule

Precautions related to the choice of a yield/flow potential:

Drucker-Prager criterion :

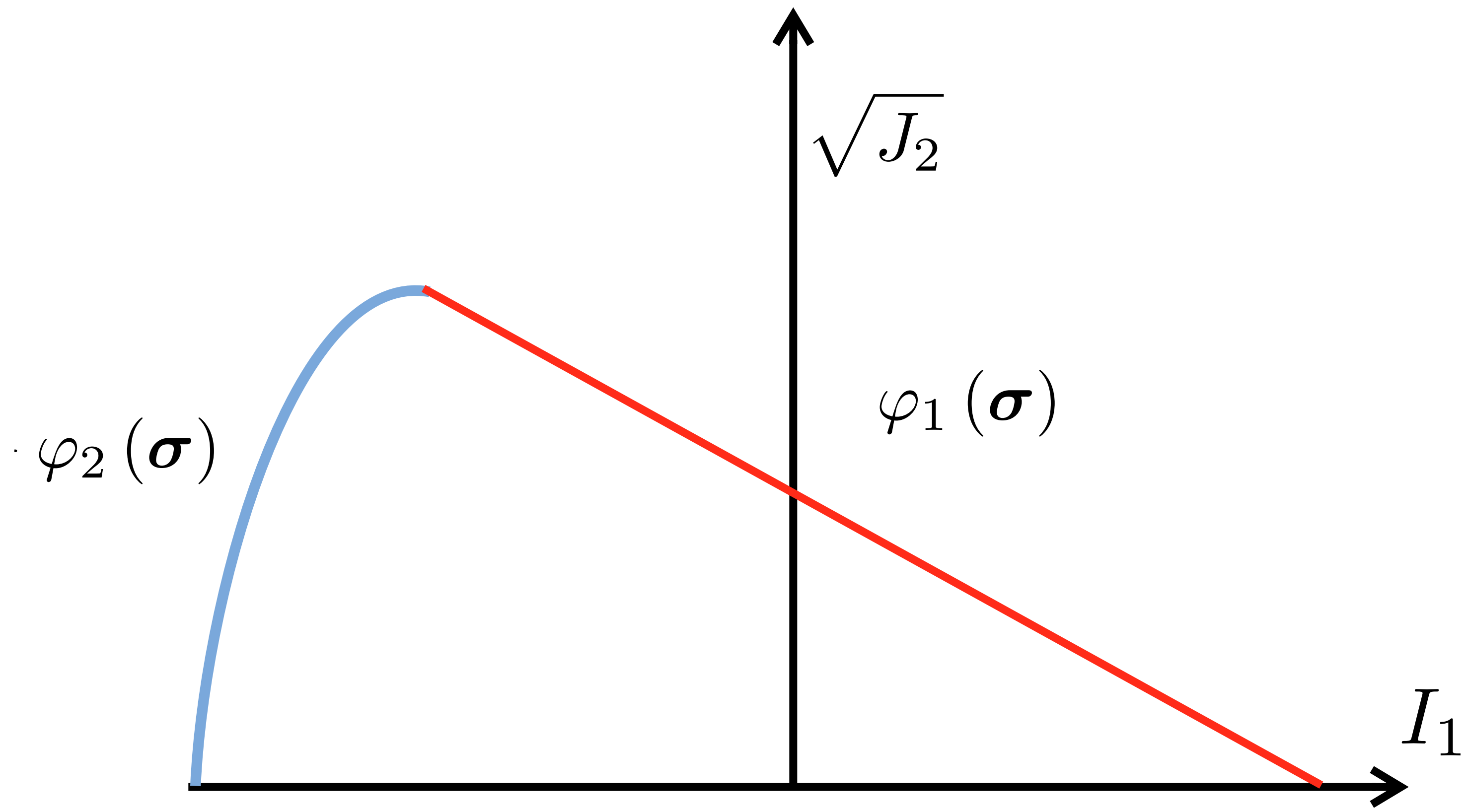
$$\varphi(\boldsymbol{\sigma}) = \sqrt{J_2} - \alpha I_1$$

$$\frac{\partial \varphi(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = \frac{\dot{\lambda}}{2\sqrt{J_2}}\boldsymbol{\sigma}' + \alpha \mathbf{I}$$

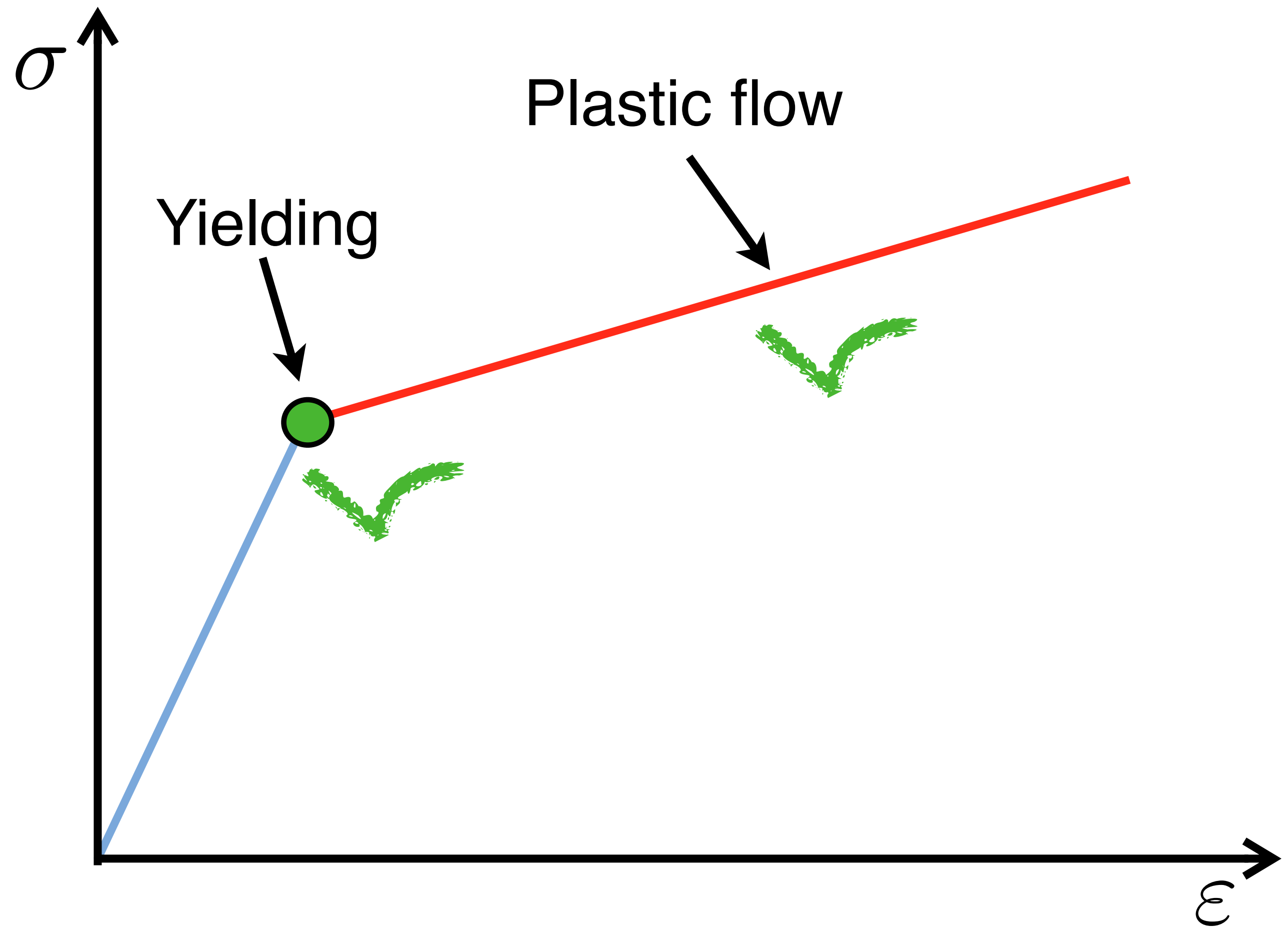


Flow rule

Precautions related to the choice of a yield/flow potential:



Flow rule



$$f \leq 0, \dot{\lambda} \geq 0, \dot{\lambda} f = 0$$

$$\dot{\lambda} \dot{f} = 0$$

- Convexity of the yield/flow potential
- Normality rule

Project discussion

- Elasticity
- Yield criterion
- Associated/Non-associated plasticity